(1) (a) $\begin{pmatrix} A^\top A & A^\top B & A^\top C \\ B^\top A & B^\top B & B^\top C \\ C^\top A & C^\top B & C^\top C \end{pmatrix}$ and $AA^\top + BB^\top + CC^\top$

(b) The matrices $A^\top A$, $B^\top B$, and $C^\top C$ are identity matrices. The matrices $A^\top C$, $B^\top C$, and $A^\top B$ are zero.

(c) They are equal to $AA^\top, BB^\top$, and $CC^\top$, respectively, and their sum is the identity matrix. They are (orthogonal) projection matrices.

(2) (a) The functions of the form $f(x) = a$.

(b) The functions of the form $f(x) = a + b\,x + c\,\sin(x) + d\,\cos(x)$.

(c) The rank is 4.

(d) A solution would be a function $g(x) \in F$ satisfying $g'(x) = f(x)$, i.e. an antiderivative of $f(x)$.

(e) If $f(x)$ lies in the column space, then there are infinitely many solutions, because if $g(x)$ is an antiderivative, then so is $g(x) + c$ for any $c \in \mathbb{R}$.

Remark: This corresponds to the description of the nullspace given in (a).

(3) The vectors span the subspace of $\mathbb{R}^4$ cut out by the equation $x_1 + x_2 + x_3 + x_4 = 0$. The dimension of this subspace is 3, so the answer is 3.

(4) (a) All three singular values of A are nonzero, so $\mathrm{rank}(A) = 3$. This implies that A has three independent columns.

(b) One of the singular values of A is approximately zero.

(5) Yes to the first question. No to the second question.

Explanation: Every basis of S can be extended to a basis of $\mathbb{R}^3$ by adjoining a vector in $\mathbb{R}^3 \setminus S$. On the other hand, a basis for $\mathbb{R}^3$ need not contain any elements of S.

(6) (a) False. Counterexample is any $n \times m$ matrix of rank n, with $n < m$, such as $\begin{pmatrix} 1 & 1 \end{pmatrix}$.

(b) False. Counterexample is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) True. We have $\mathrm{rank}(A) = \mathrm{rank}(A^\top)$, as can be seen from taking an SVD written as $A = U\Sigma V^\top$ and comparing with the SVD given by $A^\top = V\Sigma^\top U^\top$.

(d) False. Counterexample is any $n \times m$ matrix with rank $< m$, such as $\begin{pmatrix} 1 & 1 \end{pmatrix}$.

(7) All $n \times n$ matrices. All $n \times n$ matrices if $n > 1$, and $\{(0)\}$ if $n = 1$. All $n \times n$ upper triangular matrices.

Explanation: To see that any $n \times n$ matrix A is a sum of two invertible matrices, take a (full) SVD written as $A = U\Sigma V^\top$. Express $\Sigma = \Psi_1 + \Psi_2$ where Ψ_1 and Ψ_2 are diagonal matrices whose diagonal entries are all nonzero. Then Ψ_1 and Ψ_2 are invertible, so $A = U\Psi_1 V^\top + U\Psi_2 V^\top$ is a sum of two invertible matrices.

In the Lecture 9 notes, we saw that any $n \times n$ matrix can be expressed as a sum of at most n rank 1 matrices. If $n > 1$, then any rank 1 matrix is singular, so this shows that all $n \times n$ matrices can be expressed as sums of singular matrices. If $n = 1$, then the only singular matrix is (0).

The set of $n \times n$ upper triangular matrices is already a vector subspace (by a previous homework), and the span of the elements of any vector subspace is the vector subspace itself.

(8) (a) If $\left(\begin{array}{c|c} A & b \end{array} \right)$ is invertible, then its columns must be linearly independent. Therefore, b does not lie in the span of the columns of A. This implies that $Ax = b$ has no solution.

(b) If $\begin{pmatrix} A \mid b \end{pmatrix}$ is singular, it must have a nonzero nullspace. Let $y = (y_1, y_2, y_3, y_4, y_5)$ be a nonzero vector in the nullspace. If $y_5 = 0$, then (y_1, y_2, y_3, y_4) is nonzero and lies in the nullspace of A, contradicting the assumption that $\operatorname{rank}(A) = 4$. Therefore $y_5 \neq 0$. By scaling the vector y, we may assume that $y_5 = -1$. Then $x := (y_1, y_2, y_3, y_4)$ is a solution to $Ax = b$.

(9) Yes. Proof: If A_1 and A_2 are 3×3 matrices which lie in this set, and $\lambda_1, \lambda_2 \in \mathbb{R}$, then

$$(\lambda_1 A_1 + \lambda_2 A_2)\begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda_1 A_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2 A_2 \begin{pmatrix}1\\1\\1\end{pmatrix}$$

is a sum of multiples of $\begin{pmatrix}1\\1\\1\end{pmatrix}$, so it is a multiple of $\begin{pmatrix}1\\1\\1\end{pmatrix}$, so $\lambda_1 A_1 + \lambda_2 A_2$ lies in the set. Therefore, this set is a vector subspace of the vector space of all 3×3 matrices.

(10) A basis for the first vector space is $\{1, x, x^2, x^3\}$.

A basis for the second vector space is $\{x - 1, (x-1)^2, (x-1)^3\}$.