18.06 Professor Edelman Quiz 3 May 3, 2019

Your name is: ______________________________

(We need your name on every page for gradescope.)

(Note pencil can create some problems showing up in gradescope.)

Please circle your recitation:

(1)	T 10	24-307	S. Makarova
(2)	T 10	4-261	Z. Remscrim
(3)	T 11	24-307	S. Makarova
(4)	T 11	4-261	C. Hewett
(5)	T 12	4-261	C. Hewett
(6)	T 12	2-105	A. Ahn
(7)	T 1	4-149	A. Ahn
(8)	T 1	2-136	S. Turton
(9)	T 2	2-136	K. Choi
(10)	T 3	2-136	K. Choi

Important Instructions: We will be using Gradescope. Please write on one side only of a page. If you need extra pages, please write continued, and on the extra pages clearly label with problem number and letter. Note: we deleted the boxes seen on earlier tests.

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1 (25 pts.)

The matrix S has a full SVD computed with Julia, where

$$S = \begin{pmatrix} 4 & 3 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{pmatrix}.$$

There are exactly three (positive) singular values:

$$\sigma_1 = 7.9487708193188125 \; \sigma_2 = 2.5425353720085857 \; \sigma_3 = 0.593764552689772.$$

The singular vectors are the columns of

$$U = \begin{pmatrix} -0.735907 & 0.34763 & 0.065278 & 0.57735 \\ -0.481518 & -0.179604 & 0.634468 & -0.57735 \\ -0.254389 & 0.527234 & -0.56919 & -0.57735 \\ -0.402328 & -0.754268 & -0.518856 & 0 \end{pmatrix} \text{ and}$$

$$V = \begin{pmatrix} -0.735907 & -0.34763 & 0.065278 & 0.57735 \\ -0.481518 & 0.179604 & 0.634468 & -0.57735 \\ -0.254389 & -0.527234 & -0.56919 & -0.57735 \\ -0.402328 & 0.754268 & -0.518856 & 0 \end{pmatrix}.$$

1.(a) (5 pts.) What is the sum of the eigenvalues of S?

Since S is symmetric, it is diagonalizable. Namely, $S = X\Lambda X^{-1}$ where the diagonal entries of the diagonal matrix Λ are eigenvalues. Therefore, the sum of the eigenvalues is 6 by the following computation.

$$\text{trace}(\Lambda) = \text{trace}(X^{-1}X\Lambda) = \text{trace}(X\Lambda X^{-1}) = \text{trace}(A) = 4 + 2 + 0 + 0 = 6.$$

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1. (b) (5 pts.) What is the product of the eigenvalues of S? (Hint: This is easy if you look at the problem correctly and time consuming if you do not.)

Since the rank of S is 3, we have $\dim(\text{null}(S)) = 4-3 = 1 > 0$. Thus, there exists a non-zero null vector $v \in \text{null}(S)$, namely $Sv = \vec{0} = 0v$. Hence, 0 is an eigenvalue of S. In conclusion, the product of the eigenvalues is 0.

1. (c) (10 pts.) Ideally without much computation, what is the fourth column of the cofactor matrix of S?

If one deletes the fourth column and any row, one has column 1 = column 2 + column 3 in the 3x3 matrix, hence each 3x3 determinant is 0. Namely, the fourth column of the cofactor matrix is 0.

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1. (d) (5 pts.) Circle the four eigenvalues of S and justify briefly each of your choices:

$$\sigma_1, \quad \sigma_2, \quad \sigma_3, \quad -\sigma_1, \quad -\sigma_2, \quad -\sigma_3, \quad 1, \quad -1, \quad 0, \quad \sigma_1^2, \quad \sigma_2^2, \quad \sigma_3^2.$$

(Checking Suggestion: compare with your answer in 1.(a).)

Since it is a symmetric matrix, the eigenvalues are plus or minus singular values. Hence, $\{\sigma_1, -\sigma_2, \sigma_3, 0\}$ is the only case to make $\sum_{i=1}^{4} \lambda_i = 6$.

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2 (25 pts.)

2. (a) (6 pts.)

A matrix has all of its eigenvalues 1 . The set of matrices similar is

i)infinite **ii)possibly finite** iii)must be finite. (Pick and justify the best answer)

The similar matrix of I must be I itself, because $BIB^{-1} = BB^{-1} = I$. So, in this case the set is finite. However, the matrix $A = [1\,1; 0\,1]$ has infinitely many similar matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

for all $t \neq 0$.

2. (b) (6 pts.)

A matrix has all of its eigenvalues real and has a real orthogonal eigenvector matrix. True or false, this matrix must be symmetric. (Pick and justify the best answer.)

True. Suppose that S has an eigen factorization $X\Lambda X^{-1}$, and X is an orthogonal matrix, namely $X^T = X^{-1}$. Then,

$$S^T = (X\Lambda X^{-1})^T = (X\Lambda X^T)^T = (X^T)^T \Lambda^T X^T = X\Lambda X^T = X\Lambda X^{-1} = S.$$

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2. (c) (6 pts.)

The trace of the n by n reflection matrix $I - 2\frac{xx^T}{x^Tx}$ is

i) n ii) n+1 iii) n-1 **iv) n-2**. (Pick and justify the best answer.)

Solution 1: the (i,j)-th entry of xx^T is x_ix_j. Hence, $\text{trace}(xx^T) = \sum_{i=1}^n x_i^2 = x^Tx$. Therefore,

$$\text{trace}(I - 2\tfrac{xx^T}{x^Tx}) = \text{trace}(I) - \tfrac{2}{x^Tx}\text{trace}(xx^T) = n - 2.$$

Solution 2: we know from class the matrix has one eigenvalue -1 and the rest 1, so we have $(n-1) - 1 = n - 2$.

2. (d) (7 pts.)

A projection matrix is symmetric and satisfies $P^2 = P$. The possible eigenvalues of $\exp(P)$ are

i) 0 and 1.

ii) -1,0, and 1

iii) 0 and e

iv) 1 and e. (where e=exp(1)).

(Pick and justify the best answer)

Since P is symmetric, it has an eigen fatorization $P = X\Lambda X^{-1}$. Thus,

$$\Lambda = X^{-1}(X\Lambda X^{-1})X = X^{-1}PX = X^{-1}P^2X = X^{-1}(X\Lambda X^{-1})(X\Lambda X^{-1})X = \Lambda^2.$$

Hence, each diagonal entry λ_i of Λ satisfies $\lambda_i = \lambda_i^2$, namely $\lambda_i \in \{0, 1\}$.

$$\exp(P) = \sum_{k=0}^{\infty} \frac{1}{k!}P^k = \sum_{k=0}^{\infty} \frac{1}{k!}X\Lambda^kX^{-1} = X\big(\sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k\big)X^{-1} = X\exp(\Lambda)X^{-1}.$$

In addition,

$$\exp(\Lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\text{diag}[\lambda_1 \cdots \lambda_n]^k = \text{diag}[\sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k \cdots \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k] = \text{diag}[\exp(\lambda_1) \cdots \exp(\lambda_n)].$$

Namely, $\exp(\Lambda)$ is a diagonal matrix whose diagonal entries are $\exp(\lambda_i) \in \{e^0, e^1\} = \{1, e\}$.

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3 (10 pts.)

Suppose that a matrix A is not the zero matrix but A^4 is the zero matrix in 3(a) and 3(b).

3. (a) (5 pts.) The possible eigenvalues of A are

i) 0 and 1

ii) all 0

iii) 1,-1,$i, -i$

iv) 0,1,-1

(Pick and justify the best answer.)

Let λ, v be an eigenvalue and an eigenvector, namely $Av = \lambda v$ with $v \neq \vec{0}$. Then,

$$\vec{0} = A^4 v = A^3(Av) = A^3(\lambda v) = \lambda A^3 v = \cdots = \lambda^4 v,$$

implies $\lambda^4 = 0$. Thus, all eigenvalues must be 0.

3. (b) (5 pts.) The matrix A is diagonalizable. (Pick the best answer, no justification being asked for. Just right or wrong.)

i. True because the zero matrix is diagonalizable.

ii. True because A has distinct eigenvalues.

iii. False because A does not have distinct eigenvalues.

iv. False because if A were diagonalizable, and $A^4 = 0$ then A would have to be the zero matrix.

Suppose $A = X\Lambda X^{-1}$ where Λ is diagonalized. Then, $0 = X^{-1}A^4X = \Lambda^4$ implies $\Lambda = 0$. Therefore, $A = X0X^{-1} = 0$ which contradicts to the given condition $A \neq 0$.

Remind that (iii) would not be a good reason, because a matrix with repeated eigenvalues can be diagonalized.

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4 (20 pts.)

It is possible to buy a 4x4x4 Rubik's cube. This puzzle has 24 solid cubes called an edge piece. An edge piece has two colored stickers exposed. In the figure edge piece 1 and also edge piece 5 has a sticker on the top face and another sticker on the front face. The edge pieces on the front face are labeled 1 through 8 and are in white font.

4. (a) (10 pts) Write an 8x8 permutation matrix that describes the permutation of the 8 front face edge pieces when the front face is rotated clockwise 90 degrees. Label the exact meaning of P_{ij}.

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

$P_{ij} = 1$, if the edge piece j is in the position of the edge piece i after the 90 degrees clockwise rotation. Otherwise, $P_{ij} = 0$.

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4. (b) (10 pts) Consider the 24x24 permutation matrix that describes the permutation of all 24 edge pieces on the puzzle. Suppose, we make 2019 random quarter turns of faces around the puzzle. What is the determinant of this matrix? Explain your reasoning. (Hint: be sure to account for the eight pieces that move with each quarter turn of a face.)

The permutation P in 4(a) satisfies $\det P = (-1)^6 = 1$, because we can obtain I_8 by swapping rows of P six times. In the same manner, 24×24 permutation matrices of quarter turns can be I_{24} by swapping rows six times. Hence, they have determinant 1. Let us denote the 2019 random quarter turn permutations $P_1, \cdots, P_{2019}$. Then, the determinant of $\prod_{i=1}^{2019} P_i$ is

$$\det(\prod_{i=1}^{2019} P_i) = \prod_{i=1}^{2019} \det(P_i) = \prod_{i=1}^{2019} 1 = 1.$$

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5 (20 pts.)

The unit simplex in n dimensions is the following set: $\{x \in R^n$ with $0 \leq x_i \leq 1$ and $x_1 + x_2 + \ldots + x_n \leq 1\}$ This set has volume (area, etc.) $1/n!$. Suppose an $n \times n$ matrix A has determinant $d > 0$.

5. (a) (10 pts.). Consider the image of the simplex when we multiply by A. This is the set $\{Ax \in R^n \text{with } 0 \leq x_i \leq 1 \text{ and } x_1 + x_2 + \ldots + x_n \leq 1\}$. What is the volume (area, etc.) of this set?

By the volume formula in lecture slides, the volume is $\frac{1}{n!} \det A = \frac{d}{n!}$.

5. (b) (10 pts.). Consider the image of the simplex when we multiply by A^{-1}. This is the set $\{A^{-1}x \in R^n$ with $0 \leq x_i \leq 1$ and $x_1 + x_2 + \ldots + x_n \leq 1\}$. What is the volume (area, etc.) of this set?

Since $\det A = d > 0$, A is invertible and $\det A^{-1} = 1/d$. Hence, the volume is $\frac{1}{d(n!)}$.

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5 (c). (Extra Credit 5 pts.) This problem is only worth five points. Some of you may see the answer right away, but others may not see it at all. We do not recommend looking at this problem unless you have extra time, as the five points may not be worth the time lost.

Suppose the eigenvalues of A in problem 5 are $\lambda_1, \ldots, \lambda_n$. What is the volume (area, etc.) of the image of the simplex under the map $A^2 + I$, that is the set $\{(A^2 + I)x$ for $x \in R^n$ with $0 \le x_i \le 1$ and $x_1 + x_2 + \ldots + x_n \le 1\}$. Write your answer as an absolute value of an expression in terms of the eigenvalues. Explain briefly your answer.

By using $A = X\Lambda X^{-1}$, we have

$$A^2 + I = (X\Lambda X^{-1})(X\Lambda X^{-1}) + XIX^{-1} = X(\Lambda^2 + I)X^{-1}.$$

Since $\Lambda^2 + I$ is a diagonal matrix whose diagonal entries are $\lambda_i^2 + 1$, we have

$$\det(A^2 + I) = \det(X(\Lambda^2 + I)X^{-1}) = \det X \det(\Lambda^2 + I) \det X^{-1} = \det(\Lambda^2 + I) = \prod_{i=1}^{n}(\lambda_i^2 + 1).$$

Hence, the volume is $\frac{1}{n!}\prod_{i=1}^{n}(\lambda_i^2 + 1)$.

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