18.06 Professor Edelman Quiz 2 April 5, 2019

Solution

(1)	T 10	24-307	S. Makarova
(2)	T 10	4-261	Z. Remscrim
(3)	T 11	24-307	S. Makarova
(4)	T 11	4-261	C. Hewett
(5)	T 12	4-261	C. Hewett
(6)	T 12	2-105	A. Ahn
(7)	T 1	4-149	A. Ahn
(8)	T 1	2-136	S. Turton
(9)	T 2	2-136	K. Choi
(10)	T 3	2-136	K. Choi

1 (25 pts.)

A random 4x3 matrix A has a full SVD computed with Julia. The singular values are

$$2.067989079857846, \; 0.9964025831888096, \; 0.4854455453874191.$$

The singular vectors are the columns of

$$U = \begin{pmatrix} -0.534606 & 0.697017 & 0.396747 & 0.266373 \\ -0.324715 & -0.691464 & 0.539027 & 0.354805 \\ -0.650464 & -0.156256 & -0.10825 & -0.735365 \\ -0.430874 & -0.107832 & -0.735066 & 0.512247 \end{pmatrix} \text{ and}$$

$$V = \begin{pmatrix} -0.391685 & 0.466488 & 0.793077 \\ -0.729853 & 0.367332 & -0.576525 \\ -0.560265 & -0.804647 & 0.196589 \end{pmatrix}.$$

1.(a) (5 pts.) Is A^TA invertible? Why or why not?

Solution. We will denote the 4×3 matrix of singular values by Σ, with diagonal entries σ_1, σ_2, σ_3, so $A = U\Sigma V^T$. Note that we can write $A^TA = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T$, and because U is orthogonal, we have $A^TA = V\Sigma^T\Sigma V^T$. Consider the product $\Sigma^T\Sigma$. It is a product of a 3×4 matrix by a 4×3 matrix, hence is 3×3, and we can note that it is a diagonal matrix with entries σ_1^2, σ_2^2, σ_3^2 on the diagonal. These entries are nonzero, by assumption, so $\Sigma^T\Sigma$ is nonsingular. Therefore, A^TA is also nonsingular, since it is the product of three nonsingular matrices V, $\Sigma^T\Sigma$ and V^T, in other words, it is invertible.

1. (b) (5 pts.) Find and circle a vector perpendicular to every column of A.

$$U = \begin{pmatrix} -0.534606 & 0.697017 & 0.396747 & 0.266373 \\ -0.324715 & -0.691464 & 0.539027 & 0.354805 \\ -0.650464 & -0.156256 & -0.10825 & -0.735365 \\ -0.430874 & -0.107832 & -0.735066 & 0.512247 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.391685 & 0.466488 & 0.793077 \\ -0.729853 & 0.367332 & -0.576525 \\ -0.560265 & -0.804647 & 0.196589 \end{pmatrix}$$

Solution. Column space of A is spanned by the first $r = 3$ columns of U, and U is orthogonal, so its fourth column will be perpendicular to the column space of A. Answer: circle the fourth column of U.

1. (c) (5 pts.) Is AA^T invertible? Why or why not?

Solution. Similarly to part (a), we write $AA^T = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$. Note now that $\Sigma\Sigma^T$ is a 4×4 diagonal matrix with three nonzero diagonal entries, hence it is singular, so the whole product also has determinant zero and therefore is not invertible.

1. (d) (5 pts.) How many solutions to $Ax = b$ are there for a randomly generated b such as

$$b = \begin{pmatrix} 1.2616743877482997 \\ -0.6492300349356348 \\ -1.8681658666758472 \\ -1.6717363181989333 \end{pmatrix}.$$ Explain briefly.

Solution. Question about the existence of solution to $Ax = b$ is equivalent to asking whether b is in the column space of A. Column space of A is a three-dimensional hyperplane in $\mathbb{R}^4$, and a random vector does not lie there. So we should expect there to be no solutions.

1. (e) (5 pts.) What is the dimension of the orthogonal complement of the row space of A? Explain briefly.

Solution. Row space of A is of dimension $r = 3$, and is contained in $\mathbb{R}^3$, so it coincides with $\mathbb{R}^3$. But the orthogonal complement to the ambient space is just zero, so dimension of the orthogonal complement of the row space is zero.

2 (20 pts.)

2. (a) (10 pts.) Suppose $A = QR$ is the QR factorization of A into orthogonal times upper triangular. It is possible to write $\pm\det(A)$ as an expression in terms of the entries of R. What is this expression?

Solution. Recall that $\det Q = \pm 1$ and use multiplicativity of determinant to get: $\det A = \det(QR) = \det Q \det R = \pm \det R$. Now recall that determinant of an upper-triangular matrix is the product of its diagonal elements. Denoting the matrix elements of R by R_{ij}, and assuming that A is $n \times n$ (it is square, because determinant is only defined for square matrices), we get $\pm \det A = \det R = \prod_{i=1}^{n} R_{ii}$.

2. (b) (10 pts.) A fact that you can assume is that a reflector has determinant -1. Suppose you are told that Q can be written as the product of exactly 17 different reflectors, what can we say exactly about $\det(Q)$?

Solution. Denote the reflectors by R_i, for i from 1 to 17. Then $\det R_i = -1$ by assumption, and therefore $\det Q = \det(R_1 \cdots R_{17}) = \det R_1 \cdots \det R_{17} = (-1)^{17} = -1$.

3 (25 pts.) Am I a linear transformation? Briefly explain why or why not.

a. $x \to x^T Ax$ where $x \in R^n$ and A is a fixed $n \times n$ matrix.

Solution. No, because T does not respect scaling by $\lambda \in \mathbb{R}$: indeed, $T(\lambda x) = \lambda x^T A \lambda x = \lambda^2 x^T Ax = \lambda^2 T(x)$, and this is not equal to $\lambda T(x)$ for any nonzero vector x and any $\lambda \neq 0, 1$.

b. $A \to x^T Ax$ where A is an $n \times n$ matrix and $x \in R^n$ is a fixed vector.

Solution. Yes, because we can check additivity and scaling axioms:

- $T(A + B) = x^T(A + B)x = x^T(Ax + Bx) = x^T Ax + x^T Bx = T(A) + T(B)$;
- $T(\lambda A) = x^T \lambda Ax = \lambda x^T Ax = \lambda T(A)$.

c. $P(x) \to P'(x)$ where $P(x)$ is a polynomial.

Solution. Yes, because taking derivative is linear.

d. $A \to \det(A)$ where A is an $n \times n$ matrix.

Solution. If $n = 1$, then it is linear, because it is just identity. If $n > 1$, then is is not linear, because det is linear in rows, so if we scale A by, say, a factor of 2, then $\det 2A = 2^n \det A$, for example $\det 2I = 2^n \neq 2 = 2 \det I$.

e. $x \to \text{prod}(x)$ where $x \in R^n$ (and prod means compute the product of the entries in x.)

Solution. If $n = 1$, then it is linear, because it is just identity. If $n > 1$, then it is not linear, because for a vector of ones, we get $T\left(2(1, \ldots, 1)^T\right) = 2^n \neq 2 = 2T\left((1, \ldots, 1)^T\right)$.

4 (15 pts.) The $3x3$ upper triangular matrices U form a six dimensional vector space. So do the symmetric $3x3$ matrices S.

4. (a) (5 pts.) How many parameters are needed to specify a linear transformation from the $3x3$ upper triangular matrices to the symmetric $3x3$ matrices?

Solution. In order to specify a linear transformation from a 6-dimensional space to a 6-dimensional space, we need $6 \cdot 6 = 36$ parameters.

4. (b) (10 pts.) Give two different examples of linear transformations (other than the zero transformation) from the $3x3$ upper triangular matrices to the symmetric $3x3$ matrices.

Solution. Valid examples:

1. $A \mapsto A + A^T$;
2. Send a matrix to the diagonal matrix with the same diagonal: $\begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \mapsto \begin{pmatrix} a & & \\ & d & \\ & & f \end{pmatrix}$.

5 (15pts.)

Consider the nonlinear matrix function $f(A) = A^T A$. It is possible to write df as a linear transformation of dA. What is that linear transformation?

Solution. We can solve it by applying formal rules: $\mathrm{d}f(A) = \mathrm{d}(A^T A) = (\mathrm{d}A)^T A + A^T \mathrm{d}A$. Or using the notion of small increments (note that we neglect the terms that have second or higher order in $\mathrm{d}A$):

$$\begin{aligned} \mathrm{d}f(A) = f(A + \mathrm{d}A) - f(A) = (A + \mathrm{d}A)^T (A + \mathrm{d}A) - A^T A = \\ = A^T A + (\mathrm{d}A)^T A + A^T \mathrm{d}A + (\mathrm{d}A)^T \mathrm{d}A - A^T A = (\mathrm{d}A)^T A + A^T \mathrm{d}A. \end{aligned}$$

We cannot simplify further, because $(\mathrm{d}A)^T A$ is not equal to $A^T \mathrm{d}A$, for matrix multiplication is not commutative.

6. (Extra Credit 5 pts.) This problem is only worth five points. Some of you may see the answer right away, but others may not see it at all. We do not recommend looking at this problem unless you have extra time, as the five points may not be worth the time lost.

We have two matrices A and B:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}.$$

Write the products AB and BA without mechanical operations. For the five points, explain briefly your answer with words not mechanics. Graders will give points subjectively for really good expositions only.

Solution. Left multiplication by A is the operation of adding the first row scaled by a to the second row. Similarly, left multiplication by B is the operation of adding the second row scaled by b to the third row.

So if we write AB, we can think of it in terms of how it operates on matrices (in particular, vectors) on the right. So we first apply B, that is add a multiple of the second row to the third, and then add a multiple of the first row to the second, so the product is $AB = \begin{pmatrix} 1 & & \\ a & 1 & \\ 0 & b & 1 \end{pmatrix}$.

When we write it in the other order BA, we first apply A and add a multiple of the first row to the second. But then, B adds a multiple the modified second row to the third. In this case, we also have a scaled by a first row as an additional summand, so the composition works as follows: it adds the scaled by a first row to the second, the scaled by b row (of the original matrix) to the third, and the scaled by ab first row to the third. Therefore, the product is: $BA = \begin{pmatrix} 1 & & \\ a & 1 & \\ ab & b & 1 \end{pmatrix}$.