18.06 Professor Edelman Quiz 1 - Solutions March 1, 2019

1 (20 pts.)

The full SVD of

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 5 & 12 & 2 \end{pmatrix}$$

is numerically computed with Julia to be

$$U = \begin{pmatrix} -0.203600 & -0.585801 & -0.784465 \\ -0.543021 & -0.599144 & 0.588348 \\ -0.814662 & 0.545769 & -0.196116 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 6.136942826453964 & & \\ & 0.7740001393771697 & \\ & & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.365991 & 0.446524 & 0.816497 \\ -0.912869 & -0.00172137 & -0.408248 \\ -0.180887 & -0.89477 & 0.408248 \end{pmatrix}$$

1. (a) (5 pts.) The rank of this matrix is

1. (b) (5 pts.) The column space is a linear combination of some vectors found in the svd. Circle these vectors.

1. (c) (5 pts.) Circle the numbers that would figure in the best rank 1 approximation to A.

1. (d) (5 pts.) Circle the non-zero numbers that would figure in the compact (rank-r) svd

Solution.

1. (a) The rank of this matrix is $\boxed{2}$, since there are two non-zero singular values

1. (b) The column space is a linear combination of the first two columns of U, i.e. a linear combination of

$$\boxed{u_1 = \begin{pmatrix} -0.203600 \\ -0.543021 \\ -0.814662 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -0.585801 \\ -0.599144 \\ 0.545769 \end{pmatrix}}$$

1. (c) The best rank 1 approximation to A is $\sigma_1 u_1 v_1^T$, where

$$\boxed{\sigma_1 = 6.136942826453964, \quad u_1 = \begin{pmatrix} -0.203600 \\ -0.543021 \\ -0.814662 \end{pmatrix}; \quad v_1 = \begin{pmatrix} -0.365991 \\ -0.912869 \\ -0.180887 \end{pmatrix}}$$

1. (d) The compact svd for A is $A = U_1 \Sigma_r V_1^T$, where

$$U_1 = \begin{pmatrix} -0.203600 & -0.585801 \\ -0.543021 & -0.599144 \\ -0.814662 & 0.545769 \end{pmatrix}, \quad \Sigma_r = \begin{pmatrix} 6.136942826453964 & 0 \\ 0 & 0.7740001393771697 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} -0.365991 & 0.446524 \\ -0.912869 & -0.00172137 \\ -0.180887 & -0.89477 \end{pmatrix}.$$

2. (20 pts.)

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is known to have a decomposition of the form

$$A = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix},$$

where $r > 0$. Find r in terms of possibly a, b, c, d.

Solution.

We wish to write

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \\ &= \begin{pmatrix} p\sin\theta + q\cos\theta & -p\cos\theta + q\sin\theta \\ r\cos\theta & r\sin\theta \end{pmatrix} \end{aligned}$$

Comparing the bottom row of the left and right hand sides allows us to deduce that

$$c = r\cos\theta$$

$$d = r\sin\theta$$

Squaring both of these equations and adding together yields $r^2 = c^2 + d^2$, and since $r > 0$, we have that

$$\boxed{r = \sqrt{c^2 + d^2}}$$

3. (25 pts.)

Am I a vector space? Briefly explain why or why not. (Remember the zero must be in the vector space, and all real linear combinations must be in the vector space.)

a) The vectors in R^3 where $x^2 + y^2 + z^2 \leq 1$.

b) All of R^3 except those vectors along the x-axis with $x > 0$. (This means $(x, 0, 0)^T$ is excluded if $x > 0$.)

c) All 2×3 matrices whose 6 elements sum to 6.

d) All 3×3 rank 1 matrices and the 3x3 zero matrix.

e) All functions of two variables $f(x, y)$ of the form $f(x, y) = ax^2 + bxy + c$ such that $f(18.06, 2019) = 0$.

Solution.

a) This is **not** a vector space. The vector $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ satisfies this condition, but $2x$ does not. This set is therefore not closed under scalar multiplication.

b) This is **not** a vector space. The vector $x = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ satisfies this condition, but $-x$ does not. This set is therefore not closed under scalar multiplication.

c) This is **not** a vector space. The zero matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ does not satisfy this condition, but every vector space must contain the zero vector.

d) This is **not** a vector space. The matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are both rank 1, but their sum

$$C = A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is rank 2. This set is therefore not closed under vector addition.

e) This **is** a vector space. Consider the two functions $f_1(x, y) = a_1x^2 + b_1xy + c_1$ and $f_2(x, y) = a_2x^2 + b_2xy + c_2$, for which $f_1(18.06, 2019) = f_2(18.06, 2019) = 0$. Consider an arbitrary linear combination of these functions

$$g(x, y) = \mu f_1(x, y) + \lambda f_2(x, y) = (\mu a_1 + \lambda a_2)x^2 + (\mu b_1 + \lambda b_2)xy + (\mu c_1 + \lambda c_2).$$

So $g(x, y)$ has the correct form. Furthermore,

$$g(18.06, 2019) = \mu f_1(18.06, 2019) + \lambda f_2(18.06, 2019) = 0.$$

So any arbitrary linear combination of functions in this space will still be in this space. So this is a vector space.

4. (15 pts.)

How many parameters are there in an $n \times n$ "anti-symmetric" matrix? An anti-symmetric matrix has $A^T = -A$, for example when $n = 1, 2, 3$, anti-symmetric matrices look like

$$\begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Your answer should be a simple function of n

Solution.

An $n \times n$ matrix has n^2 components. In order to uniquely specify an anti-symmetric matrix, we only need to specify the entries below the main diagonal. Since there are n elements along the diagonal of a matrix, there will be

$$\boxed{\frac{n^2 - n}{2}}$$

elements below the main diagonal. This is then the number of parameters needed to specify an anti-symmetric matrix.

5. (20 pts.)

Let

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(a) (10pts.) We will inform you that the nullspace of the matrix A is a line. You should be able to tell us exactly which line this is without any difficult computations. Please describe the line.

(b) (10 pts.) The QR factorization of A^T happens to be very easy to find without any difficult computations. Write down the QR factorization of A^T.

Solution.

(a) Recall that the nullspace of a matrix is the set of all vectors for which $Ax = 0$. If we know that the nullspace is a line, then all vectors in the nullspace are parallel to each other. The nullspace condition tells us that

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.$$

All solutions to this equation are parallel to the vector

$$\boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

(b) We want to write $A^T = QR$, where $Q^T Q = I$ and R is square and upper triangular.

Notice that the columns of A^T are all mutually orthogonal:

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

However, the magnitude of each column is $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. If we were to divide each column by 2, the resulting matrix would then be orthogonal i.e.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = 1/2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

So

$$\boxed{Q = 1/2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.}$$

6. (Extra Credit 5 pts.)

This problem is only worth five points. Some of you may see the answer right away, but others may not see it at all. We do not recommend looking at this problem unless you have extra time, as the five points may not be worth the time lost.

Suppose an $n \times 2$ matrix A is written as QR, where Q is tall-skinny orthogonal and is also $n \times 2$, and

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

What is the norm of the second column of A? Explain briefly.

Solution.

Let the columns of A be the vectors $a_1, a_2 \in \mathbb{R}^n$ and let the columns of Q be the orthonormal vectors $q_1, q_2 \in \mathbb{R}^n$. The form of R tells us that

$$a_2 = q_1 + q_2.$$

We can then calculate the norm (magnitude/length) of a_2 as follows:

$$\|a_2\|^2 = a_2^T a_2 = (q_1 + q_2)^T (q_1 + q_2) = q_1^T q_1 + q_1^T q_2 + q_2^T q_1 + q_2^T q_2.$$

However, q_1 and q_2 are orthonormal, so $q_1^T q_1 = q_2^T q_2 = 1$ and $q_1^T q_2 = q_2^T q_1 = 0$. We can therefore deduce that

$$\boxed{\|a_2\| = \sqrt{2}.}$$