18.06 Professor Edelman Final Exam May 21, 2019

**Your name is:** ____________________

(We need your name on every page for gradescope.)

(Note pencil can create some problems showing up in gradescope.)

**Please circle your recitation:**

| | | | |
|---|---|---|---|
| (1) | T 10 | 24-307 | S. Makarova |
| (2) | T 10 | 4-261 | Z. Remscrim |
| (3) | T 11 | 24-307 | S. Makarova |
| (4) | T 11 | 4-261 | C. Hewett |
| (5) | T 12 | 4-261 | C. Hewett |
| (6) | T 12 | 2-105 | A. Ahn |
| (7) | T 1 | 4-149 | A. Ahn |
| (8) | T 1 | 2-136 | S. Turton |
| (9) | T 2 | 2-136 | K. Choi |
| (10) | T 3 | 2-136 | K. Choi |

Important Instructions: We will be using Gradescope. Please write on one side only of a page. If you need extra pages, please write continued, and on the extra pages clearly label with problem number and letter. Note: we deleted the boxes seen on earlier tests.

Problem Point Values: 1)16 2)11 3)17 4)12 5)12 6)10 7)4 8)10 9)8 Total:100

(sorry, no extra credit this time)

Name: ____________________________________

**1 (16 pts.)**

The matrix $A$ has a full SVD computed with Julia, where

$$A = \begin{pmatrix} 2 & 15 & 5 & 0 & 16 \\ 2 & 16 & 4 & 2 & 12 \\ 4 & 39 & 1 & 18 & -4 \end{pmatrix}.$$

The result is $A = U\Sigma V^T$ where

$$U = \begin{pmatrix} -0.341643 & -0.713606 & -0.611593 \\ -0.362087 & -0.500572 & 0.786334 \\ -0.867279 & 0.490095 & -0.0873704 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 48.46518677202946 & 0 & 0 & 0 & 0 \\ 0 & 21.520354810093167 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$V = \begin{pmatrix} -0.10062 & -0.0217458 & 0.852961 & 0.292097 & -0.420167 \\ -0.923177 & 0.0186082 & 0.0386047 & -0.380465 & 0.0339893 \\ -0.0830254 & -0.236066 & 0.309187 & 0.29878 & 0.867475 \\ -0.33705 & 0.363403 & -0.343575 & 0.793231 & -0.0841173 \\ -0.130861 & -0.900773 & -0.239433 & 0.226805 & -0.25043 \end{pmatrix}$$

1.(a) (4 pts.) We wish to know which vectors in U and V correspond to the nullspace, the column space, the left nullspace, and the row space. Draw a box or circle around the vectors in each of these subspaces and label your box/circle.

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1. (b) (2 pts.) What is the rank $r$ of $A$?

1. (c) (5 pts.) Find the complete solution to $Ax = u_2$ where $u_2$ is the second column of $U$. (Okay to use symbols such as $\sigma_1$, $u_1$, or $v_1$. Be sure to have the complete solution, not just a solution.)

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1. (d) (5 pts.) Ideally without precious time wasting computation, what is the determinant of the first three columns (reproduced below) of $A$? Justify your answer.

$$\begin{pmatrix} 2 & 15 & 5 \\ 2 & 16 & 4 \\ 4 & 39 & 1 \end{pmatrix}$$

Name: ____________________________________

**2 (11 pts.)** For 2(a) and 2(b) compute the gradient with respect to $A$ (ideally without matrix elements or indices). It may be helpful to remember that trace($X$)= trace($X^T$)=trace($X^T I$) and trace ($XY$)=trace($YX$).

2. (a) (3 pts.) $f(A) = \text{trace}(\frac{1}{2}A^T A)$, where the matrices are $n \times n$.

2. (b) (3 pts.) $f(A) = \text{trace}(A)$.

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2. (c) (5 pts.) Compute $d(\exp .(A^3))$ in terms of $A$ and $dA$. The dot indicates that we are taking the exponential $e^x$ of every entry (not the matrix exponential). $A^3$ is the usual matrix multiplication of $A$ times $A$ times $A$.

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**3 (17 pts.)**

Set up an idealized version of Bluebikes with bicycle stations in Allston, Boston, and Cambridge where on any given day, a bicycle has a 40% chance of remaining in the same city every night. There is a 30% chance of going to each of the other two cities. (Thus for example a bicycle starting in Cambridge has a 30% chance of ending up in Boston and a 30% chance of ending up in Allston.)

3. (a) (5 pts.) Write down the relevant Markov Matrix.

3. (b) (2 pts.) Is your matrix in 3(a) diagonalizible?

Name: ______________________________

3. (c) (5 pts.) What are the eigenvalues of your matrix in 3(a)?

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3. (d) (5 pts.) Suppose that on the first day 90% of the bicycles are in Cambridge, 1% in Allston, and 9% in Boston. What percentage of bicycles would you estimate would be in Cambridge once the bicycles have reached steady state?

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**4 (12 pts.)**

Give a brief and convincing argument for each statement. (Not an example.)

4. (a) (3 pts.) The $2019^{\text{th}}$ power of a symmetric matrix is symmetric.

4. (b) (3 pts.) The $2019^{\text{th}}$ power of a positive definite matrix is positive definite (at least from the pure mathematics viewpoint).

4. (c) (3 pts.) The $2019^{\text{th}}$ power of a permutation matrix is a permutation matrix.

Name: ______________________________

4. (d) (3 pts.) The $2019^{\text{th}}$ power of an orthogonal matrix is orthogonal.

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**5 (12 pts.)**

The $5 \times 5$ matrix $A$ has a QR decomposition where $R$ is the diagonal matrix with $1, 2, 3, 4, 5$ on the diagonal, and 0 off the diagonal.

5. (a) (3 pts.) If $a_5$ is the fifth column of $A$, what is $\|a_5\|$?

5. (b) (3 pts.) If $a_4$ is the fourth column of $A$, what is the inner product of $a_4$ and $a_5$. (Reminder this means compute $a_4^T a_5$) Explain your answer.

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5. (c) (3 pts.) The linear transformation that takes $x$ to $Ax$ takes the five dimensional unit cube to a parallelopiped. What is the unsigned volume of the image of the five dimensional unit cube? (Remember this means the absolute value of the volume of the parallelopiped determined by the columns of $A$.)

5. (d) (1 pt.) Pick the best one (without explanation) of {must be, might be, can't be}: The matrix $A$ ____________ orthogonal.

5. (e) (1 pt.) Pick the best one (without explanation) of {must be, might be, can't be}: The matrix $A$ ____________ (symmetric) positive definite.

5. (f) (1 pt.) Pick the best one (without explanation) of {must be, might be, can't be}: The columns of $Q$ ____________ semi-axes of the ellpsoid that is the image of the unit sphere under $A$.

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**6 (10 pts.)**

6. (a) (6 pts.) Set up a matrix least squares problem if we are interested in taking n data points $(x_i, y_i)$ for $i = 1, \ldots, n$ and we wish to find the best function $f(x) = c_1 \sin(x) + c_2 \cos(x) + c_3 \tan(x)$ through the data points. Note: Setting up a matrix least squares problem means setting up a matrix $A$ and a vector $b$ in a least squares equation such that $x = \hat{x}$ minimizes $\|Ax - b\|$.

Also write the solution to the least squares problem in terms of the compact SVD of your matrix.

6. (b) (4 pts.) Set up a matrix least squares problem if we are interested in taking $n$ data points $(x_i, y_i, z_i)$ in $R^3$, and we wish to fit a function $f(x, y) = c_1 e^{x+y} + c_2 \sin(x - y)$.

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**7 (4 pts.)** The 4x4 symmetric matrix $A$ satisfies $A^2 = A$ and has all four diagonal elements equal to 1/2.

The four roots to the equation $\det(A - \lambda I) = 0$ are not distinct. Allowing for multiple eigenvalues, they are __________, __________, __________, and __________. (Provide a brief explanation.)

Name: ______________________________

**8 (10 pts.)**

There are six permutation matrices in $R^3$. Let $P_1, P_2, P_3$ be the three with determinant -1 (labelled arbitrarily.) The other three are $I$ and $P_4$ and $P_5$.

8 (a) (4 pts.) Give an example of one permutation matrix with determinant -1, and one that is not the identity but has determinant +1.

8. (b) (6 pts.) Consider the matrix $1000P_1 + 800P_3 + 6P_5$. Give an eigenvalue, eigenvector pair for this matrix (ideally by not writing down the matrix).

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**9 (8 pts.)**

Given $n$ numbers $h_1, \ldots, h_n$, a Hankel matrix is defined as a matrix $H$ such that $H_{ij} = h_{|i-j|+1}$. They take the form

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & \ldots & h_{n-1} & h_n \\ h_2 & h_1 & h_2 & \ldots & h_{n-2} & h_{n-1} \\ h_3 & h_2 & h_1 & \ldots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ h_{n-1} & h_{n-2} & h_{n-3} & \ldots & h_1 & h_2 \\ h_n & h_{n-1} & h_{n-2} & \ldots & h_2 & h_1 \end{pmatrix}.$$

(a) (4 pts.) Are the set of Hankel matrices a vector subspace of symmetric $n \times n$ matrices? (Explain)

(b) (4 pts.) Find a basis for the vector space of Hankel matrices. What is the dimension?

Enjoy your upcoming summer!

Name (if page needed): ____________________

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