PROBLEM SET VII

DUE THURSDAY, 5 MAY 2016

(1) Using only the identities $e^{i\theta} = \cos\theta + i\sin\theta$ and $\cos^2\theta + \sin^2\theta = 1$ along with the basic functioning of the complex numbers, prove the following trigonometric identities:
  (a) $\cos(\alpha \pm \beta) = (\cos\alpha)(\cos\beta) \mp (\sin\alpha)(\sin\beta)$;
  (b) $\sin(\alpha \pm \beta) = (\sin\alpha)(\cos\beta) \pm (\cos\alpha)(\sin\beta)$;
  (c) $\sin^2\alpha = \frac{1}{2}(1 - \cos(2\alpha))$;
  (d) $\cos^2\alpha = \frac{1}{2}(1 + \cos(2\alpha))$.
  (In the same way, you can also prove a formula for $\cos^n$ and $\sin^n$ in terms of only sines and cosines.)

(2) For any complex number $z = a + bi$, consider the matrix

$$M_z \coloneqq \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Prove that these matrices contain all the algebraic properties of $\mathbf{C}$ by verifying the following:

(a) $M_0 = 0$ and $M_1 = I$;
(b) $M_{z+w} = M_z + M_w$ for any $z, w \in \mathbf{C}$;
(c) $M_{-z} = -M_z$ for any $z \in \mathbf{C}$;
(d) $M_{zw} = M_z M_w$ for any $z, w \in \mathbf{C}$;
(e) $M_{z^{-1}} = M_z^{-1}$ for any $z \in \mathbf{C}$ such that $z \neq 0$;
(f) $M_{\overline{z}} = M_z^{\intercal}$ for any $z \in \mathbf{C}$;
(g) $|z|^2 = \det M_z$ for any $z \in \mathbf{C}$;
(h) $t^2 - (z + \overline{z})t + z\overline{z} = p_{M_z}(t)$ for any $z \in \mathbf{C}$;
(i) $M_{\rho \exp(i\theta)} = \rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for any $\rho \geq 0$ and any $\theta \in [0, 2\pi)$.

(3) Consider the $n \times n$ matrix

$$A = \begin{pmatrix} \hat{e}_2 & \cdots & \hat{e}_n & \hat{e}_1 \end{pmatrix}.$$

Compute the characteristic polynomial and the complex eigenvalues of A. Is A diagonalizable over **R**? over **C**?

(4) Suppose $\theta \in [0, 2\pi)$. What are the complex eigenvalues and correpsonding complex eigenspaces of the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}?$$

(5) Suppose $\widehat{x} \in \mathbf{C}^n$ a vector such that $\widehat{x}^* \widehat{x} = 1$. What are the eigenspaces of $I - 2\widehat{x}\widehat{x}^*$?

(6) Suppose A is an $n \times n$ matrix with characteristic polynomial

$$p_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 x + a_0.$$

Find an expression for $p_{A^{-1}}(t)$ by contemplating the determinant of $(tI - A^{-1})A$.

(7) For any $n \geq 2$, give an example of an invertible $n \times n$ matrix that is not diagonalizable over $\mathbf{C}$.