SOLUTIONS TO PROBLEM SET 6

18.06 SPRING 2016

Note the difference of conventions: these solutions adopt that the characteristic polynomial of a matrix A is $\det(A - xI)$ while the lectures adopt the convention that it is $\det(tI - A)$. The difference between the two is the sign $(-1)^n$. As far as the answers are concerned, it only affects problem 1.

(1) *What is the constant term of the characteristic polynomial of a square matrix? Why?*

The constant term of a polynomial $P(x)$ is its value when $x = 0$. By definition of the characteristic polynomial, its value when $x = 0$ is the determinant of the matrix.

Answer: The determinant of the matrix.

(2) *Compute the eigenvalues of the matrix*

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Compute the characteristic polynomial

$$\det\begin{pmatrix} -x & -1 & 0 & 0 \\ -1 & -x & -1 & 0 \\ 0 & -1 & -x & -1 \\ 0 & 0 & -1 & -x \end{pmatrix} = -x \cdot \det\begin{pmatrix} -x & -1 & 0 \\ -1 & -x & -1 \\ 0 & -1 & -x \end{pmatrix} - (-1) \cdot \det\begin{pmatrix} -1 & -1 & 0 \\ 0 & -x & -1 \\ 0 & -1 & -x \end{pmatrix} =$$

$$= -x\left(-x \cdot \det\begin{pmatrix} -x & -1 \\ -1 & -x \end{pmatrix} - (-1) \cdot \det\begin{pmatrix} -1 & -1 \\ 0 & -x \end{pmatrix}\right) - (-1) \cdot (-1) \cdot \det\begin{pmatrix} -x & -1 \\ -1 & -x \end{pmatrix} =$$

$$= -x(-x(x^2 - 1) + x) - (x^2 - 1) = x^4 - 3x^2 + 1.$$

The eigenvalues are the roots of this polynomial. Let $y = x^2$. Then y is a root of the quadratic polynomial $y^2 - 3y + 1$, and $y \geq 0$. Solving the quadratic polynomial, we get $y = \frac{3 \pm \sqrt{5}}{2}$. Both roots are positive, so given that $x = \pm\sqrt{y}$ we get four possibilities for x.

Answer: $\sqrt{\frac{3+\sqrt{5}}{2}}, \sqrt{\frac{3-\sqrt{5}}{2}}, -\sqrt{\frac{3+\sqrt{5}}{2}}, -\sqrt{\frac{3-\sqrt{5}}{2}}$.

Or alternatively: $\frac{\pm 1 \pm \sqrt{5}}{2}$ (note that $\frac{1+\sqrt{5}}{2} = \sqrt{\frac{3+\sqrt{5}}{2}}$ etc.).

(3) *The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?*

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}$$

For a matrix A, the eigenspace with eigenvalue λ is the kernel of the matrix $A-\lambda I$. Here we have $\lambda = 1$, so we subtract I from each of the matrices above:

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\1&0&0\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$$

and find the dimensions of the kernels.

The ranks of these matrices are 0, 2, 2, 1 respectively, so by the rank-nullity theorem the dimensions of the kernels are 3, 1, 1, 2.

Answer: 3, 1, 1, 2.

(4) *Is the matrix*

$$\begin{pmatrix}1&1&0\\1&1&1\\0&1&1\end{pmatrix}$$

diagonalizable?

Short solution: Yes, because it is symmetric. Symmetric matrices are always diagonalizable.

Long solution: An $n \times n$ matrix is diagonalizable if and only if the dimensions of its eigenspaces add up to n. Let us find the eigenvalues and then find the dimension of the eigenspace for each eigenvalue.

To find the eigenvalues, let us write the characteristic polynomial.

$$\det\begin{pmatrix}1-x&1&0\\1&1-x&1\\0&1&1-x\end{pmatrix} = (1-x)\cdot\det\begin{pmatrix}1-x&1\\1&1-x\end{pmatrix} - 1\cdot\det\begin{pmatrix}1&1\\0&1-x\end{pmatrix} =$$

$$= (1-x)((1-x)^2-1) - (1-x) = (1-x)((1-x)^2-2) = (1-x)(x^2-2x-1).$$

The roots of the quadratic polynomial x^2-2x-1 are $\frac{2\pm\sqrt{8}}{2} = 1\pm\sqrt{2}$, therefore the eigenvalues of the matrix are 1, $1+\sqrt{2}$, $1-\sqrt{2}$. We have three distinct eigenvalues, and each of them has an eigenspace of dimension at least 1, so the sum of the dimensions is at least 3. On the other hand, in a 3-dimensional space this sum is at most 3, so it equals 3, so the matrix is diagonalizable.

(Note that by the same argument an $n \times n$ matrix with n distinct eigenvalues is always diagonalizable).

Answer: Yes.

(5) *If n is odd, then every $n \times n$ matrix has at least one eigenvector in $\mathbb{R}^n$. Why?*

The degree of the characteristic polynomial of an $n \times n$ matrix equals n. If n is odd, a polynomial of degree n always has a real root.

(6) *Suppose $n \geq 2$, and consider the $n \times n$ matrix $A = (\alpha_{i,j})$ whose entries are given by*

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } j = i+1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) *Write a formula for the entries of the matrix A^k for $0 \leq k \leq n$.*
(b) *For $1 \leq k \leq n$, compute the eigenvalues and eigenspaces of A^k.*

Let e_i be the ith vector of the standard basis. Then $Ae_1 = 0$, $Ae_2 = e_1$, $Ae_3 = e_2$ etc. Therefore, $A^2 e_i = 0$ for $i = 1, 2$ and $A^2 e_i = e_{i-2}$ for $i \geq 3$. Similarly, $A^k e_i = 0$ for $i = 1, \dots, k$, and $A^k e_i = e_{i-k}$ for $k < i \leq n$. Using this, we can write a formula for the entries of A^k: we have $(A^k)_{i,j} = 1$ if $j = i + k$ and 0 otherwise. Note that this also covers $k = 0$, when $A^0 = I$.

For the eigenvectors of the matrix A^k, note that the vectors $e_1, \dots, e_k$ are eigenvectors with eigenvalue 0, so any vector of the form $\sum_{i=1}^{k} a_i e_i$ is an eigenvector with eigenvalue 0. Suppose there is some other eigenvector $v = \sum_{j=1}^{n} b_j e_j$, and let $m > k$ be the highest index with $b_m \neq 0$ (we can tell that $m > k$ from the assumption that v is not a linear combination of $e_1, \dots, e_k$). Then $A^k v = \sum_{j=1}^{m} b_j A^k e_j = \sum_{j=k+1}^{m} b_j e_{j-k}$, in particular, the component along e_m equals 0. Therefore, v can only be an eigenvector if its eigenvalue is 0. On the other hand, $b_m e_{m-k} \neq 0$, so eigenvalue 0 is not a possibility either. We conclude that there are no eigenvectors other than the linear combinations of $e_1, \dots, e_k$.

Answer: a) $(A^k)_{i,j} = 1$ if $j = i + k$ and 0 otherwise; b) the only eigenvalue of A^k is 0, and the space of eigenvectors is the span of $e_1, \dots, e_k$.

(7) *Suppose $\widehat{x} \in \mathbb{R}^n$ a unit vector. Recall from Exam III the Householder matrix $H = I - 2\widehat{x}\widehat{x}^T$ and the hyperplane*

$$N := \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \widehat{x} = 0\}$$

(which is the orthogonal complement to $\widehat{x}$).

(a) *If you werent able to show that for any $\vec{w} \in \mathbb{R}^n$, one has $\pi_N(\vec{w}) = \pi_N(H\vec{w})$ on Exam III, please write up a proof here in your own words!*
(b) *Prove that for any $\vec{w} \in \mathbb{R}^n$, one also has*

$$\vec{w} - \pi_N(\vec{w}) = \pi_N(H\vec{w}) - H\vec{w}.$$

Explain what H does geometrically; draw a picture for $n = 2$ and $n = 3$.
(c) *Purely from geometry, compute the eigenvalues and eigenspaces of H. (You dont have to compute any determinants for this.) Is H diagonalizable?*

The $n \times n$ matrix $\widehat{x}\widehat{x}^T$ is the matrix of the projection of the space $\mathbb{R}^n$ onto the line spanned by the vector $\widehat{x}$. (Note: this is not to be confused with $\widehat{x}^T\widehat{x}$, which is a number equal to $\widehat{x} \cdot \widehat{x}$). Denote this projection by π_x. The line spanned by $\widehat{x}$ and the hyperplane N are orthogonal complements, so for any $\vec{w} \in \mathbb{R}^n$ we have $\vec{w} = \pi_N(\vec{w}) + \pi_x(\vec{w})$. The vector $\pi_N(\vec{w}) = \vec{w} - \pi_x(\vec{w})$ is the projection of $\vec{w}$ onto N; the vector $H\vec{w} = \vec{w} - 2\pi_x(\vec{w})$ is the reflection of $\vec{w}$ with respect to N. In other words, $H\vec{w} = \pi_N(\vec{w}) - \pi_x(\vec{w})$.

Now it is clear that the vectors $\vec{w}$ and $H\vec{w}$, its reflection with respect to N, have the same projection onto N (part (a)). Moreover, $\vec{w} - \pi_N(\vec{w}) = \pi_x(\vec{w}) = -\pi_x(H\vec{w}) = \pi_N(H\vec{w}) - H\vec{w}$ (part (b)). Finally, as a reflection, H has eigenvalues 1 and -1. The eigenspace for eigenvalue 1 is the hyperplane N, and the eigenspace for eigenvalue -1 is the line spanned by $\widehat{x}$. Since their dimensions add up to n, the matrix H is diagonalizable (part (c)).

(8) *For every permutation $\sigma \in \Sigma_3$, compute the eigenvalues and eigenspaces of the 3×3 matrix P_σ.*

For $\sigma = \mathrm{id}$, we have $P_\sigma = I$, so the only eigenvalue is 1 and the corresponding eigenspace is $\mathbb{R}^3$.

The two matrices corresponding to the two cycles of length 3 are rotations around the line $x = y = z$, so the only eigenvalue is 1, and the corresponding eigenspace is the line $x = y = z$.

The three transpositions correspond to reflections, so the eigenvalues are -1 (with multiplicity 1) and 1 (with multiplicity 2).

Answer: (let the coordinates on $\mathbb{R}^3$ be x, y, z, in this order; permutations given in cycle notation).

- $\sigma = \mathrm{id}$, $P_\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Eigenvalue 1: eigenspace $\mathbb{R}^3$.
- $\sigma = (12)$, $P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Eigenvalue 1: eigenspace $x - y = 0$ (a plane), eigenvalue -1: eigenspace $(x + y = 0,\ z = 0)$ (a line).
- $\sigma = (13)$, $P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Eigenvalue 1: eigenspace $x - z = 0$ (a plane), eigenvalue -1: eigenspace $(x + z = 0,\ y = 0)$ (a line).
- $\sigma = (23)$, $P_\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Eigenvalue 1: eigenspace $y - z = 0$ (a plane), eigenvalue -1: eigenspace $(y + z = 0,\ x = 0)$ (a line).
- $\sigma = (123)$, $P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Eigenvalue 1: eigenspace $x = y = z$ (a line).
- $\sigma = (132)$, $P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Eigenvalue 1: eigenspace $x = y = z$ (a line).

(9) *Does the matrix*

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

have any real eigenvalues?

Compute the characteristic polynomial:

$$\det \begin{pmatrix} 1-x & -1 & 0 & 0 \\ 1 & 1-x & -1 & 0 \\ 0 & 1 & 1-x & -1 \\ 0 & 0 & 1 & 1-x \end{pmatrix} = (1-x)\cdot\det \begin{pmatrix} 1-x & -1 & 0 \\ 1 & 1-x & -1 \\ 0 & 1 & 1-x \end{pmatrix} - (-1)\cdot\det \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1-x & -1 \\ 0 & 1 & 1-x \end{pmatrix} =$$

$$= (1-x)\left((1-x)\cdot\det\begin{pmatrix} 1-x & -1 \\ 1 & 1-x \end{pmatrix} - (-1)\cdot\det\begin{pmatrix} 1 & -1 \\ 0 & 1-x \end{pmatrix}\right) + \det\begin{pmatrix} 1-x & -1 \\ 1 & 1-x \end{pmatrix} =$$

$$= (1-x)((1-x)((1-x)^2+1) + 1 - x) + (1-x)^2 + 1.$$

Let $t = 1 - x$. Then the polynomial takes form $t(t(t^2+1)+t)+t^2+1 = t^4+3t^2+1$. This polynomial has no real roots since both t^2 and t^4 are always non-negative, so neither does the characteristic polynomial of the matrix. Therefore, this matrix has no real eigenvalues.

Answer: No.

(10) *What is the characteristic polynomial of the matrix*

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{pmatrix}?$$

From the previous problem sets we know that the rank of this matrix is 2 (we can also easily find this out by row reduction). Therefore, by the rank-nullity theorem, the kernel of this matrix has dimension 2. By definition, the kernel of a matrix is the eigenspace for eigenvalue 0, so we know that the geometric multiplicity of 0 is 2. Hence the algebraic multiplicity of 0 is at least 2, i.e. the polynomial is divisible by x^2. Therefore, it has the form $ax^4 + bx^3 + cx^2$.

Let us look at the three coefficients a, b, c. The characteristic polynomial consists of all possible terms of the following form: choose n entries in the matrix A so that the combination has exactly one entry from each row and each column; for each entry on the diagonal make an additional choice, taking either a_{ii} or $-x$; multiply all together and add sign. Among these terms, the ones which have x in degree k are of the following form: choose k entries from the diagonal (that's where your x's come from); from the matrix A, cross out the columns and rows that contain these entries; take the determinant of the $(n-k)\times(n-k)$ matrix you got; multiply by $(-x)^k$. In this manner, we see that the degree n term is always $(-1)^n x^n$, and the degree $(n-1)$ term is $(-1)^{n-1}\sum a_{ii}x^{n-1}$ (the value $\sum a_{ii}$ is also called the *trace* of the matrix).

Thus we have $a = 1$, $b = -1-2-5-13 = -21$, and

$$c = \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + \det\begin{pmatrix} 1 & 3 \\ 3 & 13 \end{pmatrix} + \det\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} + \det\begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix} + \det\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} =$$
$$= 1+1+4+1+1+1 = 9.$$

Answer: $x^4 - 21x^3 + 9x^2$.