PROBLEM SET VI

DUE THURSDAY, 21 APRIL 2016

(1) What is the constant term of the characteristic polynomial of a square matrix? Why?

(2) Compute the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

(3) The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(4) Is the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

diagonalizable?

(5) If n is odd, then every $n \times n$ matrix has at least one eigenvector in $\mathbf{R}^n$. Why?

(6) Suppose $n \geq 2$, and consider the $n \times n$ matrix $A = (\alpha_{i,j})$ whose entries are given by

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } j = i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write a formula for the entries of the matrix A^k for $0 \leq k \leq n$.

(b) For $1 \leq k \leq n$, compute the eigenvalues and eigenspaces of A^k.

(7) Suppose $\widehat{x} \in \mathbf{R}^n$ a unit vector. Recall from Exam III the *Householder matrix* $H = I - 2\widehat{x}\widehat{x}^\intercal$ and the hyperplane

$$N := \{\vec{v} \in \mathbf{R}^n \mid \vec{v} \cdot \widehat{x} = 0\}$$

(which is the orthogonal complement to $\widehat{x}$).

(a) If you weren't able to show that for any $\vec{w} \in \mathbf{R}^n$, one has $\pi_N(\vec{w}) = \pi_N(H\vec{w})$ on Exam III, please write up a proof here in your own words!

(b) Prove that for any $\vec{w} \in \mathbf{R}^n$, one also has

$$\vec{w} - \pi_N(\vec{w}) = \pi_N(H\vec{w}) - H\vec{w}.$$

Explain what H does geometrically; draw a picture for $n = 2$ and $n = 3$.

(c) Purely from geometry, compute the eigenvalues and eigenspaces of H. (You don't have to compute any determinants for this.) Is H diagonalizable?

Optional additional problems

(8) For every permutation $\sigma \in \Sigma_3$, compute the eigenvalues and eigenspaces of the 3×3 matrix P_σ.

(9) Does the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

have any real eigenvalues?

(10) What is the characteristic polynomial of the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{pmatrix}?$$