18.06 (Spring 16) Pset 5 solutions

Due April 13th, 2016

1. For $k > 1$, write $I - A^k = (I - A)B$, where $B = (I + A + A^2 + \cdots + A^{k-1})$. Then,

$$det(I - A^k) = det(I - A)det(B) \neq 0$$

because $I - A^k$ is invertible. Therefore, both $det(I - A) \neq 0$ and $det(B) \neq 0$, and in particular, $I - A$ is invertible.

2. From the first three rows, at most two can be linearly independent. Therefore, the matrix is non-invertible and the determinant is zero.

3. There are many ways of computing this determinant. One way is to perform some elementary row operations to transform the matrix into an upper triangular one, by keeping the determinant invariant. For instance,

$$R_6 \leftarrow R_6 - 3R_1$$

$$R_5 \leftarrow R_5 - 3R_2$$

$$R_4 \leftarrow R_4 - 3R_3$$

We are left with $det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} = 1 \cdot 1 \cdot 1 \cdot (-2) \cdot (-2) \cdot (-2) = -8.$

4. We can easily check for $n = 1$ and $n = 2$ that $d_1 = 1$ and $d_2 = det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$.

The last column of any A_n is of the form $(0, \ldots, 0, -1, 1)$. Expanding the determinant d_n for the last column, we get

$$d_n = d_{n-1} - det \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}_{n-1 \times n-1}$$

Expanding in terms of the last column again, we end up with

$$d_n = d_{n-1} + d_{n-2}$$

as desired.

5. (a) First, let's prove the existence. Define the vector $x = (x_i) \in \mathbb{R}^n$ by:

$$x_i = det(v_1, v_2, \dots, v_{n-1}, e_i)$$

For any $w \in \mathbb{R}^n$, write $w = \sum w_i e_i$ and one has:

$$x \cdot w = \sum x_i w_i = \sum w_i det(v_1, v_2, \dots, v_{n-1}, e_i)$$

Using the multilinearity property of determinants

$$x \cdot w = det(v_1, v_2, \dots, v_{n-1}, \sum w_i e_i) = det(v_1, v_2, \dots, v_{n-1}, w)$$

as desired.

Now, let's show the uniqueness. Assume there exists a different vector $y = (y_i) \in \mathbb{R}^n$ such that for any $w \in \mathbb{R}^n$, one has

$$y \cdot w = det(v_1, v_2, \dots, v_{n-1}, w)$$

In particular for $w = e_i$,

$$y_i = y \cdot e_i = det(v_1, v_2, \dots, v_{n-1}, e_i) = x_i$$

Therefore, $x = y$.

(b) In 5a), we can use $w = x$ to get

$$||x||^2 = x \cdot x = det(v_1, v_2, \dots, v_{n-1}, x) = det(A)$$

where A is the matrix with columns $v_1, v_2, \dots, v_{n-1}, x$. Using properties of determinants

$$det(A)^2 = det(A)det(A) = det(A^T)det(A) = det(A^T A)$$

where $A^T A$ is an $n \times n$ matrix given by

$$A^T A = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_{n-1} & v_1 \cdot x \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_{n-1} & v_2 \cdot x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & v_{n-1} \cdot v_2 & \cdots & v_{n-1} \cdot v_{n-1} & v_{n-1} \cdot x \\ x \cdot v_1 & x \cdot v_2 & \cdots & x \cdot v_{n-1} & x \cdot x \end{pmatrix}.$$

Note that by 5a), the cross product x is orthogonal to all v_i, for $i = 1, \dots, n-1$, $x \cdot v_i = det(v_1, v_2, \dots, v_{n-1}, v_i) = 0$, so $A^T A$ simplifies to

$$A^T A = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_{n-1} & 0 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_{n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & v_{n-1} \cdot v_2 & \cdots & v_{n-1} \cdot v_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & x \cdot x \end{pmatrix}.$$

Combining everything:

$$||x||^4 = det(A)^2 = det(A^T A) = ||x||^2 det(M)$$

where M is the $(n-1) \times (n-1)$ matrix whose (i,j)-th entry is $v_i \cdot v_j$. Thus, $||x|| = \sqrt{det(M)}$.