Correction Pset 4, 18.06

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Problem 1

The Gram-Schmidt procedure does not modify orthormal sets of vectors $\{\vec{v_1}, \dots, \vec{v_k}\}$ in $\mathbb{R}^n$. Let us prove this statement by induction on k. If $k = 1$ it is obvious. Consider a orthonormal set $\{\vec{v_1}, \dots, \vec{v_{k+1}}\}$ of $k+1$ vectors in $\mathbb{R}^n$. We first apply GS to the set $\{\vec{v_1}, \dots, \vec{v_k}\}$. By the induction hypothesis, this set stays unchanged. Then we compute the orthogonal projection of $\vec{v_{k+1}}$ onto the space spanned by $\vec{v_1}, \dots, \vec{v_k}$. Since $\vec{v_{k+1}}$ is orthogonal to $\vec{v_i}$ for $i = 1 \dots k$, this projection is $\vec{0}$. Therefore the GS procedure does not change the set $\{\vec{v_1}, \dots, \vec{v_{k+1}}\}$.

Problem 2

The matrices $\Pi(n)$ are symmetric therefore we only need to compute Ker $\Pi(n)$ and Im $\Pi(n)$. If we call C_i^n the i-th column of the matrix $\Pi(n)$ we have for all $n \geq 4$ and $i \geq 4$

$$C_i^n = C_{i-2}^n + C_{i-3}^n \tag{1}$$

Therefore, when $n \geq 4$, the column space of $\Pi(n)$ is the span of the first three columns. We can check that these three first columns are independent. Hence, C_1^n, C_2^n, C_3^n is a basis of Im $\Pi(n)$ when $n \geq 4$. By the rank-nullity theorem, we have when $n \geq 4$:

$$dim \ \text{Ker}\,\Pi(n) = n - 3$$

To find a basis of Ker $\Pi(n)$, it suffices to exhibit $n-3$ independent vectors in Ker $\Pi(n)$. We deduce them from equation (1). They are given by, for $i \geq 4$

$$v_i = e_i - e_{i-2} - e_{i-3}$$

Finally, it is straighforward to check that $\Pi(1)$, $\Pi(2)$ and $\Pi(3)$ are invertible matrices. Hence they have trivial kernels and their image is $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Problem 3

After 3 elementary row operations we can bring the matrix A to the following matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 & -8 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix}$$

where

$$\text{Ker}(A) = \text{Ker}(B)$$

Therefore a basis of Ker A is given by $\vec{v_1} = (-1, 3, -3, 1, 0)^T$ and $\vec{v_2} = (-3, 8, -6, 0, 1)^T$. Let us orthogonalize $\{\vec{v_1}, \vec{v_2}\}$. We have $\pi_{\vec{v_2}}(\vec{v_1}) = \frac{45}{20}\vec{v_1}$, hence let $\vec{w_2} = \vec{v_2} - \pi_{\vec{v_2}}(\vec{v_1}) = \frac{1}{4}(-3, 5, 3, -9, 4)^T$. Then $\{\vec{v_1}, \vec{w_2}\}$ is a orthogonal basis of Ker A. We have

$$\pi_{\text{Ker A}}(\vec{b}) = \pi_{\vec{v_1}}(\vec{b}) + \pi_{\vec{w_2}}(\vec{b}) = -\frac{2}{5}(-1, 3, -3, 1, 0)^T + \frac{2}{35}(-3, 5, 3, -9, 4)^T = \frac{8}{35}(1, -4, 6, -4, 1)^T$$

Problem 4

Let $(e_i)_{i=1\dots n}$ be the canonical basis of $\mathbb{R}^n$. We define the vectors $v_i \in \mathbb{R}^n$ for $i = 1 \dots n$ by

$$v_i = (1, \dots, 1)^T - e_i$$

It is easy to see that $v_j.v_j = n - 1$ and that $v_i.v_j = n - 2$ when $i \neq j$. Let $\{u_1, \dots, u_n\}$ be the GS orthogonalization of $\{v_1, \dots, v_n\}$. Then $u_1 = v_1$ and u_k $(k \geq 2)$ is of the form

$$u_k = v_k + c_k^1 v_1 + \cdots + c_k^{k-1} v_{k-1}$$

where $c_k^1, \dots, c_k^{k-1}$ are real numbers. The GS process indicates that u_k is orthogonal to $v_1, \dots, v_{k-1}$, i.e.,

$$u_k.v_j = (v_k + c_k^1 v_1 + \cdots + c_k^{k-1} v_{k-1}).v_j = 0$$

for $j = 1, \dots, k-1$. By symmetry we know that $c_k^1 = \cdots = c_k^{k-1}$ and an easy computation shows that

$$c_k^1 = \cdots = c_k^{k-1} = -\frac{n-2}{n-1+(k-2)(n-2)}$$

So

$$\begin{aligned} u_k &= v_k - \frac{n-2}{n-1+(k-2)(n-2)}(v_1 + \dots v_{k-1}) \\ &= \frac{1}{n-1+(k-2)(n-2)}(1, \dots, 1, -k(n-2), 3-n, \dots, 3-n)^T \end{aligned}$$

where 1 appears for $k-1$ times and $3-n$ appears for $n-k$ times. The GS orthonormalization of $\{v_1, \dots, v_n\}$ is given by $\{\frac{u_1}{||u_1||}, \dots, \frac{u_n}{||u_n||}\}$.

Problem 5

(a) Let $v = (v_0, \dots, v_n)^T$ and $w = (w_0, \dots, w_n)$ be two vectors in $\mathbb{R}^{n+1}$ and r, s be two real numbers. Since $rv + sw = (rv_0 + sw_0, \dots, rv_n + sw_n)$, we have by definition

$$\begin{aligned} p_{rv+sw}(x) &= \sum_{i=0}^{n} (rv_i + sw_i)x^i \\ &= r\sum_{i=0}^{n} v_i x^i + s\sum_{i=0}^{n} w_i x^i \\ &= rp_v(x) + sp_w(x) \end{aligned}$$

(b) (i) Let v and w be two vectors in $\mathbb{R}^{n+1}$. Then,

$$\begin{aligned} \langle v|w\rangle &= \int_{-1}^{1} p_v(x).p_w(x)dx \\ &= \int_{-1}^{1} p_w(x).p_v(x)dx \\ &= \langle w|v\rangle \end{aligned}$$

(ii) Let v, w and u be three vectors in $\mathbb{R}^{n+1}$ and r, s be two real numbers. We have :

$$\begin{aligned} \langle rv + sw|u\rangle &= \int_{-1}^{1} p_{rv+sw}(x).p_u(x)dx \\ &= \int_{-1}^{1} (rp_v(x) + sp_w(x)).p_u(x)dx \\ &= r\int_{-1}^{1} p_v(x).p_u(x)dx + s\int_{-1}^{1} p_w(x).p_u(x)dx \\ &= r\langle v|u\rangle + s\langle w|u\rangle \end{aligned}$$

(iii) Let v be a vector in $\mathbb{R}^{n+1}$ such that $\langle v|w\rangle = 0$ for all $w \in \mathbb{R}^{n+1}$. In particular,

$$\langle v|v\rangle = \int_{-1}^{1} p_v(x)^2 dx = 0$$

The integral of a continuous and non-negative function on an interval is trivial if and only the function itself is trivial. Therfore, $p_v(x) = 0$ on the interval $[-1, 1]$. Since a nontrivial polynomial cannot have infinitely many zeroes, we get that $p_v = 0$, hence that $v = 0$.

(c) Let $v = (1, \dots, 1)^T$. By definition, we have

$$\begin{aligned}
\langle v|v\rangle &= \int_{-1}^{1} p_v(x)^2 dx \\
&= \int_{-1}^{1} (1 + x + \cdots + x^n)^2 dx \\
&= \int_{-1}^{1} (\sum_{i,j=0}^{n} x^{i+j}) dx \\
&= \sum_{i,j=0}^{n} \int_{-1}^{1} x^{i+j} dx \\
&= \sum_{i,j=0}^{n} \frac{1 - (-1)^{i+j+1}}{i+j+1} \\
&= \sum_{k=0}^{2n} p_k . \frac{1 - (-1)^{k+1}}{k+1}
\end{aligned}$$

where p_k denotes the number of pairs (i, j) such that $i + j = k$, $0 \leq i, j \leq n$ for $k = 0, \dots, 2n$. The length of v is given by $\sqrt{\langle v|v\rangle}$.

(d) Let i and j be two integers between 0 and n.

$$\begin{aligned}
\langle e_i|e_j\rangle &= \int_{-1}^{1} p_{e_i}(x).p_{e_j}(x) dx \\
&= \int_{-1}^{1} x^i.x^j dx \\
&= \int_{-1}^{1} x^{i+j} dx \\
&= \frac{1 - (-1)^{i+j+1}}{i+j+1}
\end{aligned}$$

So e_i and e_j are orthogonal if and only if $i + j$ is odd.

(e) We have $u_0 = e_0$. By (d), $u_0.e_1 = 0$ hence $u_1 = e_1$. To compute u_2, we need to compute $u_0.e_2$ and $u_1.e_2$. The second quantity is 0 for the same reason. As for the first one, $u_0.e_2 = \frac{2}{3}$. Moreover, $u_0.u_0 = 2$. Hence $u_2 = e_2 - \frac{1}{3}e_0$. The only non-trivial term while computing the projection of e_3 into the space spanned by u_0, u_1, u_2 is $u_1.e_3 = \frac{2}{5}$. Since $u_1.u_1 = \frac{2}{3}$ we obtain $u_3 = e_3 - \frac{3}{5}e_1$. Using similar reasoning, we find $u_4 = e_4 - \frac{6}{7}u_2 - \frac{1}{5}u_0 = e_4 - \frac{6}{7}e_2 + \frac{3}{35}e_0$. In terms of polynomials,

$$\begin{aligned} p_{u_0}(x) &= 1 \\ p_{u_1}(x) &= x \\ p_{u_2}(x) &= x^2 - \frac{1}{3} \\ p_{u_3}(x) &= x^3 - \frac{3}{5}x \\ p_{u_4}(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35} \end{aligned}$$

(f) Let us denote by Q_n the polynomial $(\frac{d}{dx})^n((x^2-1)^n)$ for $n \geq 0$. Note that the degree of Q_n is n for all $n \geq 0$. Let us use the notation V_n for the space of polynomials of degree at most n. In other words, V_n is the span of $1, x, \dots, x^n$. We have a sequence of inclusions

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \dots$$

Let us prove that the degree of p_{u_n} is n for all $n \geq 0$. It is true when $n = 0$. Assume it is true up to the integer n. Since by GS the span of $u_0, \dots, u_{n+1}$ is the same as the span of the vectors $e_0, \dots, e_{n+1}$, we see that the degree of $p_{u_{n+1}}$ is at most $n+1$. If it was strictly less than $n+1$ we would have $n+2$ independent vectors $u_0, \dots, u_{n+1}$ in V_n. This is not possible because the dimension of V_n is $n+1$.

The next step is to prove that the polynomial Q_n is orthogonal to V_{n-1} for the scalar product $\langle .|. \rangle$. Let $P \in V_{n-1}$, i.e. a polynomial of degree at most $n-1$. In particular, $(\frac{d}{dx})^n(P) = 0$. We want to show that

$$\int_{-1}^{1} P.Q_n = 0$$

It is not too hard to check by induction that $(\frac{d}{dx})^j(Q_n) = R_j(x).(x^2-1)^{n-j}$ for some polynomial R_j for $j = 0, \dots, n-1$. Consequentely, $(\frac{d}{dx})^j(Q_n)(1) = (\frac{d}{dx})^j(Q_n)(-1) = 0$ for $j = 0, \dots, n-1$. Therefore, performing integration by part n times gives

$$\begin{aligned} \int_{-1}^{1} P.Q_n &= \int_{-1}^{1} P(x).(\frac{d}{dx})^n((x^2-1)^n)dx \\ &= (-1)^n \int_{-1}^{1} (\frac{d}{dx})^n(P).Q_n(x)dx \\ &= 0 \end{aligned}$$

At this point, we have enough information to claim the following fact : p_{u_n} and Q_n are proportional for all n. Remember that the u_n comes from GS applied to the canonical basis e_i. Therefore it is orthogonal to the space spanned by $e_0, \dots, e_{n-1}$ which is precisely V_{n-1}. Both p_{u_n} and Q_n are elements of the space V_n which are orthogonal to the space V_{n-1}. The dimension of the space V_n (resp. V_{n-1}) is n (resp. $n-1$). Therefore the dimension of ${V_{n-1}}^{\perp}$ is 1. The polynomials p_{u_n} and Q_n belong to a space of dimension 1, so they are proportional. We can find the coefficient of proportionnality by looking at the highest degree term of each of them. The leading term of p_{u_n} is x^n. The leading term of Q_n is $\frac{2n!}{n!}$. We can then conclude that for all $n \geq 0$,

$$p_{u_n(x)} = \frac{n!}{2n!}(\frac{d}{dx})^n(x^2-1)^n$$