**PROBLEM SET IV**

DUE THURSDAY, 7 APRIL 2016

(1) Suppose $\{\vec{v}_1, \ldots, \vec{v}_k\}$ an orthonormal set of vectors in $\mathbf{R}^n$. What happens when you apply the Gram–Schmidt process to this set? Why?

(2) The *Padovan numbers* $p(n)$ are defined in a manner similar to the Fibonacci numbers: we define

$$p(0) = p(1) = p(2) = 1,$$

and for $n \geq 3$, we set

$$p(n) \coloneqq p(n-2) + p(n-3).$$

Now consider the $n \times n$ matrix $\Pi(n)$ whose $i, j$-th entry is given by

$$p(i+j).$$

For $1 \leq n \leq 5$, find a basis for each of the four fundamental spaces: $\ker(\Pi(n))$, $\operatorname{im}(\Pi(n))$, $\operatorname{coker}(\Pi(n))$, and $\operatorname{im}(\Pi(n))$. Can you say what will happen in general?

(3) Here's a vector of $\mathbf{R}^5$:

$$\vec{b} \coloneqq \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

and here's a $3 \times 5$ matrix

$$A \coloneqq \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}.$$

Compute the projection $\pi_{\ker(A)}(\vec{b})$ of the vector $\vec{b}$ onto the subspace $\ker(A) \subset \mathbf{R}^5$.

(4) Here's a basis of $\mathbf{R}^n$:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right\}.$$

This is the basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ where

$$\vec{v}_j = \sum_{i \neq j} \hat{e}_i.$$

What is the Gram–Schmidt orthonormalization of this basis?

(5) *Challenging.* The Gram–Schmidt process isn't just for the dot product. It works equally well for more exonic inner products. Here's a fun example for you to work through.

(a) The starting place is to think of a vector

$$\vec{v} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{pmatrix} \in \mathbf{R}^{n+1}$$

as a way of encoding the coefficients of a polynomial in a variable $x$:

$$p_{\vec{v}}(x) \coloneqq \sum_{0 \leq i \leq n} \alpha_i x^i = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n.$$

Prove that for any two vectors $\vec{v}, \vec{w} \in \mathbf{R}^{n+1}$ and for any two scalars $r, s \in \mathbf{R}$, we have

$$p_{r\vec{v}+s\vec{w}}(x) = r p_{\vec{v}}(x) + s p_{\vec{w}}(x).$$

(The fancy-sounding thing to say is that $p$ defines a linear map (in fact an isomorphism) from $\mathbf{R}^{n+1}$ to the vector space of polynomials of degree $\leq n$.)

(b) Now define, for any two vectors $\vec{v}, \vec{w} \in \mathbf{R}^{n+1}$, a number

$$\langle \vec{v}, \vec{w} \rangle = \int_{-1}^{+1} p_{\vec{v}}(x) p_{\vec{w}}(x)\, dx.$$

This defines something called a *scalar product* on $\mathbf{R}^{n+1}$: in effect, you input two vectors, and you get out a real number. We want to think of this as formally analogous to the dot product. To see that analogy, check the following identities:

(i) For any two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$,

$$\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle.$$

(ii) For any three vectors $\vec{v}, \vec{w}, \vec{x} \in \mathbf{R}^n$, and for any two numbers $r, s \in \mathbf{R}$,

$$\langle r\vec{v} + s\vec{w}, \vec{x} \rangle = r\langle \vec{v}, \vec{x} \rangle + s\langle \vec{w}, \vec{x} \rangle.$$

(iii) Suppose $\vec{v} \in \mathbf{R}^n$ is a vector. If, for *every* vector $\vec{w} \in \mathbf{R}^n$, one has $\langle \vec{v}, \vec{w} \rangle = 0$, then $\vec{v} = \vec{0}$.

(c) According to this scalar product, how "long" is the vector

$$\vec{v} = (1, \ldots, 1)?$$

(d) Now we see that the standard basis $\{\hat{e}_0, \ldots, \hat{e}_n\}$ is no longer "orthogonal" with respect to this new scalar product. (Note that we're indexing things in a slightly different way, because we have $n+1$ basis vectors.) Indeed, compute, for any $1 \leq i, j \leq n$, the number

$$\langle \hat{e}_i, \hat{e}_j \rangle.$$

For which $i$ and $j$ do you get zero?

(e) Now, finally, let's apply the Gram–Schmidt orthogonalization process – with respect to this crazy new scalar product! – to the standard basis $\{\hat{e}_0, \dots, \hat{e}_n\}$. So we define, iteratively,

$$\begin{aligned}
\vec{u}_0 &= \hat{e}_0; \\
\vec{u}_1 &= \hat{e}_1 - \frac{\langle \vec{u}_0, \hat{e}_1 \rangle}{\langle \vec{u}_0, \vec{u}_0 \rangle} \vec{u}_0; \\
\vec{u}_2 &= \hat{e}_2 - \frac{\langle \vec{u}_0, \hat{e}_2 \rangle}{\langle \vec{u}_0, \vec{u}_0 \rangle} \vec{u}_0 - \frac{\langle \vec{u}_1, \hat{e}_2 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1; \\
&\vdots \\
\vec{u}_n &= \hat{e}_n - \sum_{i=0}^{n-1} \frac{\langle \vec{u}_i, \hat{e}_n \rangle}{\langle \vec{u}_i, \vec{u}_i \rangle} \vec{u}_i.
\end{aligned}$$

(We won't bother with the normalization step, because that'll just introduce a bunch of square roots no one wants.) Compute $p_{\vec{u}_i}$ for $0 \le i \le 4$.

(f) (This bit's very difficult, and totally optional.) Relate $p_{\vec{u}_n}$ to the $n$-th derivative of $(x^2 - 1)^n$.