PROBLEM SET III

DUE 7 MARCH 2016

(1) Solve the equations $A\vec{x} = \vec{0}$ (i.e., find a basis for $\ker(A)$) in the following situations.
   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
   (b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
   (c) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$

   Challenging (extra credit): can you compute the kernel of the matrix

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{pmatrix}$$

   in general?

(2) Solve the equations $A\vec{x} = \vec{0}$ (i.e., find a basis for $\ker(A)$) in the following situations.

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$

Challenging (extra credit): can you compute the kernel of

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n+1 & \cdots & 2n-1 \end{pmatrix}$$

in general?

(3) The $m \times n$ *checkerboard* matrix is the matrix A whose entries are

$$a_{ij} = \frac{1}{2}(1 + (-1)^{i+j}).$$

Find a basis for the kernel of A. (Hint: this should only require a little computation.)

(4) Suppose you have an $(m+p, n+q)$ matrix of the form

$$X = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right),$$

where A is an $m \times n$ matrix, B is an $m \times q$ matrix, C is a $p \times q$ matrix, and 0 is the $p \times n$ matrix that is all zeroes. Suppose you know the nullity of A and C. What, if anything, can you conclude about the nullity of X?

(5) Define a matrix

$$Q := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compute $Q^{-1}, Q^2, Q^{-2}, Q^3, Q^{-3}$. Write a formula for the entries of any power Q^k (positive or negative) involving *Fibonacci numbers* defined by $f_0 := 0$, $f_1 := 1$, and for $n \geq 1$,

$$f_{n+1} := f_n + f_{n-1}.$$

Prove using Q that $f_n^2 + (-1)^n = f_{n-1} f_{n+1}$. What other things can you learn by playing with Q? (This one doesn't use much from this week – all the Q^k are of course invertible – I just think it's fun.)

(6) Let F be the $m \times n$ matrix whose entries are

$$a_{ij} = f_{i+j},$$

the $(i + j)$-th Fibonacci number from the previous problem. Find a basis for the kernel of A.