PROBLEM SET II

DUE 29 FEBRUARY 2016

(1) Invert the following square matrices using whatever method you prefer. (They are all invertible!)

(a) $\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 4 & 6 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{pmatrix}$

That last one's a bit of a pain if you don't find a shortcut (there are several), but the result is pretty darn interesting!!

(2) Your roommate wakes you up at 3AM in a fit of rage, cursing that he/she has “no idea whether this [expletive deleted] matrix is invertible.” You look at the matrix he/she has drawn, and in your drowsy, bleary-eyed state, all you can tell is that it’s a 2×2 matrix with 3 positive entries and one negative entry. Can you answer your roommate so you can get back to sleep?

(3) If $1 \leq i < j \leq n$ and if r is a real number, let $L_{ij}(r)$ be the $n \times n$ matrix whose entry in row u and column v is given by

$$l_{uv} := \begin{cases} r & \text{if } u = i \text{ and } v = j; \\ 1 & \text{if } u = v; \\ 0 & \text{otherwise.} \end{cases}$$

So it's just like the identity matrix, except the (i, j)-th entry is changed to r. Matrices that look like this are sometimes called *elementary* matrices.

(a) If A is an $n \times n$ matrix, what are $L_{ij}(r)A$ and $AL_{ij}(r)$? Express your answer in terms of row and column vectors.

(b) Must the inverse of $L_{ij}(r)$ be another elementary matrix? If so, which one?

(4) Is this 18×18 matrix invertible?

$$
\begin{pmatrix}
1 & 2 & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 2 & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 3 & 4 & 0 & 0 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 0 & 4 & 0 & 0 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 8 & 0 & 0 & 0 & 0 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 0 & 0 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 8 & 0 & 10 & 0 & 0 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 0 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 14 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 14 & 15 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 14 & 0 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 14 & 15 & 16 & 17 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 14 & 15 & 16 & 0 & 18
\end{pmatrix}
$$

Explain! (You don't need to use the determinant to answer this question, but can you guess the determinant of this matrix anyhow?)

(5) Here's a system of 512 linear equations in 512 variables $x_1, x_2, \dots, x_{512}$:

$$\begin{aligned}
513 &= x_2 + x_3 + \cdots + x_{511} + x_{512};\\
514 &= x_1 + x_3 + \cdots + x_{511} + x_{512};\\
&\vdots\\
1023 &= x_1 + x_2 + \cdots + x_{510} + x_{512};\\
1024 &= x_1 + x_2 + \cdots + x_{510} + x_{511}.
\end{aligned}$$

Does it have a unique solution?