18.06 - Problem Set 1 Solutions
February 16th, 2016

Problem 1 *Are the following collections of vectors in* $\mathbf{R}^3$ *linearly independent? Why or why not?*

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.00001 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix} \right\}$

(f) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(g) $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Solution: In each case below let us refer to the collection of vectors in question as S.

(a) S is not linearly independent. Indeed, $1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$ is a nontrivial solution to $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ for $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Here $\vec{0}$ is our notation for the origin of any vector space $\mathbf{R}^n$.

(b) S is linearly independent. Indeed, suppose $\alpha_1 \begin{pmatrix}5\\2\\3\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\2\\5\end{pmatrix} = \vec{0}$. By taking the dot product of this equation with $\vec{e}_2$ we see that

$$2\alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2.$$

Then by taking the dot product with $\vec{e}_1$ we see that

$$5\alpha_1 + 3\alpha_2 = 0 \Rightarrow 2\alpha_1 = 0 \Rightarrow \alpha_1 = 0.$$

But this also means $\alpha_2 = 0$. So our solution must have been trivial.

(c) S is not linearly independent since $17 \begin{pmatrix}1\\0\\2\end{pmatrix} - 34 \begin{pmatrix}0\\0\\1\end{pmatrix} + 1 \begin{pmatrix}17\\0\\0\end{pmatrix} = \vec{0}$.

(d) S is linearly independent. Let us prove this using a slightly different technique from what we did in (b). Recall the following very important fact (let us call it the *two-out-of-three criterion*)– if T is a finite collection of vectors in $\mathbb{R}^n$ then any two of the following together imply the third:

- the number of vectors in T is n;
- the vectors in T span $\mathbb{R}^n$;
- the vectors in T are linearly independent.

So, since $\#S = 3$, we can show S is linearly independent by showing it spans $\mathbb{R}^3$. Here is another simple but useful fact: to show that T spans $\mathbb{R}^n$ it is enough to show that each standard basis vector $\vec{e}_i$ for $i = 1, 2, \ldots, n$ can be expressed as a linear combination of vectors in T. Thus to show S is linearly independent we need only show that $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ can be expressed as a linear combination of vectors in S. We can do that as follows:

$$0 \begin{pmatrix}1\\0\\2\end{pmatrix} + 0 \begin{pmatrix}0\\0.00001\\1\end{pmatrix} + \frac{1}{17} \begin{pmatrix}17\\0\\0\end{pmatrix} = \vec{e}_1;$$

$$-50000 \begin{pmatrix}1\\0\\2\end{pmatrix} + 100000 \begin{pmatrix}0\\0.00001\\1\end{pmatrix} + \frac{50000}{17} \begin{pmatrix}17\\0\\0\end{pmatrix} = \vec{e}_2;$$

$$\frac{1}{2} \begin{pmatrix}1\\0\\2\end{pmatrix} + 0 \begin{pmatrix}0\\0.00001\\1\end{pmatrix} - \frac{1}{34} \begin{pmatrix}17\\0\\0\end{pmatrix} = \vec{e}_3.$$

(e) S is linearly independent. Since $\#S = 3$, we can use follow the same approach as the last problem and establish the S is linearly independent by expressing $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as linear combinations of vectors in S, as follows:

$$\frac{14}{33}\begin{pmatrix}2\\1\\6\end{pmatrix} + \frac{3}{33}\begin{pmatrix}5\\2\\2\end{pmatrix} - \frac{10}{33}\begin{pmatrix}1\\2\\9\end{pmatrix} = \vec{e}_1$$

$$-\frac{43}{33}\begin{pmatrix}2\\1\\6\end{pmatrix} + \frac{12}{33}\begin{pmatrix}5\\2\\2\end{pmatrix} - \frac{26}{33}\begin{pmatrix}1\\2\\9\end{pmatrix} = \vec{e}_2$$

$$\frac{8}{33}\begin{pmatrix}2\\1\\6\end{pmatrix} - \frac{3}{33}\begin{pmatrix}5\\2\\2\end{pmatrix} - \frac{1}{33}\begin{pmatrix}1\\2\\9\end{pmatrix} = \vec{e}_3.$$

(f) S is linearly independent. Again since $\#S = 3$, we can establish the S is linearly independent by expressing $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as linear combinations of vectors in S, as follows:

$$-\frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{e}_1$$

$$\frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{e}_2$$

$$\frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{e}_3.$$

(g) S is linearly independent. Again since $\#S = 3$, we can establish the S is linearly independent by expressing $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as linear combinations of vectors in S, as follows:

$$0\begin{pmatrix}1\\-1\\-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\\-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\vec{e}_1$$

$$-\frac{1}{2}\begin{pmatrix}1\\-1\\-1\end{pmatrix}+0\begin{pmatrix}-1\\1\\-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\vec{e}_2$$

$$-\frac{1}{2}\begin{pmatrix}1\\-1\\-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\\-1\end{pmatrix}+0\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\vec{e}_3.$$

Problem 2 *Write, if possible, each of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbf{R}^3$ as a linear combination of the following collections of vectors. If it is not possible, explain why not.*

(a) $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

(b) $\left\{\begin{pmatrix}5\\2\\3\end{pmatrix},\begin{pmatrix}3\\2\\5\end{pmatrix}\right\}$

(c) $\left\{\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}17\\0\\0\end{pmatrix}\right\}$

(d) $\left\{\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\0.00001\\1\end{pmatrix},\begin{pmatrix}17\\0\\0\end{pmatrix}\right\}$

(e) $\left\{\begin{pmatrix}2\\1\\6\end{pmatrix},\begin{pmatrix}5\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\9\end{pmatrix}\right\}$

(f) $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$

(g) $\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$

Solution: In each case below let us refer to the collection of vectors in question as S.

(a) It is clearly not possible to express any of the vectors $\vec{e}_1$, $\vec{e}_2$, or $\vec{e}_3$ as a linear combination of vectors in S. Indeed, the set of linear combinations of vectors of S is just the point $\{\vec{0}\}$.

(b) It is not possible to express any of the vectors $\vec{e}_1$, $\vec{e}_2$, or $\vec{e}_3$ as a linear combination of vectors in S. Suppose that $\vec{e}_1$ could be written as a linear combination of vectors in S: then we have

$$\alpha_1 \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \vec{e}_1$$

for some $\alpha_1, \alpha_2 \in \mathbf{R}^2$; taking the dot product of this equation with $\vec{e}_2$ we see $\alpha_2 = -\alpha_1$; next, taking the dot product with $\vec{e}_1$ we see $5\alpha_1 + 3\alpha_2 = 1 \Rightarrow \alpha_1 = \frac{1}{2}$; and finally taking the dot product with $\vec{e}_3$ we see $3\alpha_1 + 5\alpha_2 = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \frac{1}{2} = 0$, a contradiction. So indeed $\vec{e}_1$ cannot be so expressed. Next suppose $\vec{e}_2$ could be written as a linear combination of vectors in S: then we have

$$\alpha_1 \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \vec{e}_2$$

for some $\alpha_1, \alpha_2 \in \mathbf{R}^2$; taking the dot product with $\vec{e}_1$ we see $5\alpha_1 + 3\alpha_2 = 0 \Rightarrow \alpha_2 = -\frac{3}{5}\alpha_1$; next, by taking the dot product with $\vec{e}_2$ we see that $2\alpha_1 + 2\alpha_2 = 1 \Rightarrow 2\alpha_1 - \frac{6}{5}\alpha_1 = 1 \Rightarrow \alpha_1 = \frac{5}{4}$; finally taking the dot product with $\vec{e}_3$ we see that $5\alpha_1 + 3\alpha_2 = 0 \Rightarrow 5\alpha_1 + \frac{9}{5}\alpha_1 = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \frac{5}{4} = 0$, a contradiction. So indeed $\vec{e}_2$ cannot be so expressed. Finally, suppose $\vec{e}_3$ could be written as a linear combination of vectors in S: then we have

$$\alpha_1 \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \vec{e}_3$$

for some $\alpha_1, \alpha_2 \in \mathbf{R}^2$; taking the dot product with $\vec{e}_2$ we have $2\alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_1$; next, taking the dot product with $\vec{e}_1$ we have $5\alpha_1 + 3\alpha_2 = 0 \Rightarrow 2\alpha_1 = 0 \Rightarrow \alpha_1 = 0$; finally, taking the dot product with $\vec{e}_3$ we have $3\alpha_1 + 5\alpha_2 = 1 \Rightarrow -2\alpha_1 = 1 \Rightarrow \alpha_1 = -\frac{1}{2} \Rightarrow 0 = -\frac{1}{2}$, a contradiction. So indeed $\vec{e}_3$ cannot be so expressed.

(c) We can express $\vec{e}_2$ and $\vec{e}_3$ as linear combinations of vectors in S as follows:

$$0\begin{pmatrix}1\\0\\2\end{pmatrix}+1\begin{pmatrix}0\\0\\1\end{pmatrix}+0\begin{pmatrix}17\\0\\0\end{pmatrix}=\vec{e}_2$$
$$0\begin{pmatrix}1\\0\\2\end{pmatrix}+0\begin{pmatrix}0\\0\\1\end{pmatrix}+\frac{1}{17}\begin{pmatrix}17\\0\\0\end{pmatrix}=\vec{e}_3.$$

On the other hand, we cannot express $\vec{e}_1$ as a linear combination of vectors in S. Why is this? Because if we could, then all the basis vectors of $\mathbb{R}^3$ would lie in the span of S, which would mean S would span $\mathbb{R}^3$. But by the two-out-of-three criterion, that would imply that S was linearly independent. And we have seen in Problem 1 that S is not linearly independent.

(d) It is possible to write each of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as a linear combination of vectors in S and indeed we already did this in Problem 1.

(e) It is possible to write each of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as a linear combination of vectors in S and indeed we already did this in Problem 1.

(f) It is possible to write each of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as a linear combination of vectors in S and indeed we already did this in Problem 1.

(g) It is possible to write each of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as a linear combination of vectors in S and indeed we already did this in Problem 1.

Problem 3 *How many solutions does each of the following systems of linear equations have? (Answer* without *solving them, if you can!)*

(a)

$$\begin{aligned} x+17z&=3\\ 2x+z&=0 \end{aligned}$$

(b)

$$\begin{aligned} 5x-7y+17z&=2\\ 19x+12y-9z&=88\\ -113x+y-z&=-1 \end{aligned}$$

(c)

$$\begin{aligned} x + y + 2z &= 1 \\ w + x + 2y &= 1 \\ v + w + 2x &= 1 \\ u + v + 2w &= 1 \end{aligned}$$

(d)

$$\begin{aligned} u + v + w + x + y - 2z &= 0 \\ u + v + w + x - 2y + z &= 0 \\ u + v + w - 2x + y + z &= 0 \\ u + v - 2w + x + y + z &= 0 \\ u - 2v + w + x + y + z &= 0 \\ -2u + v + q + x + y + z &= 0 \end{aligned}$$

Solution: First we make a general observation. A set $T = \{\vec{v}_1, \dots, \vec{v}_n\}$ of vectors in $\mathbf{R}^n$ satisfying any two of the two-out-of-three criterion is called a *basis*. If T is a basis, then for any $\vec{u} \in \mathbf{R}^n$ there are unique $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ such that $\vec{u} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$. Because T spans $\mathbf{R}^n$ there are certainly *some* scalars like this. Why are they unique? Suppose to the contrary that there were also $\beta_1, \dots, \beta_n \in \mathbf{R}$ with $\vec{u} = \beta_1\vec{v}_1 + \cdots + \beta_n\vec{v}_n$ and there is at least one i such that $\alpha_i \neq \beta_i$. Then by subtracting the two equations we would have $\vec{0} = (\alpha_1 - \beta_1)\vec{v}_1 + \cdots + (\alpha_n - \beta_n)\vec{v}_n$, with $(\alpha_i - \beta_i) \neq 0$, contradicting the fact that T is linearly independent. So indeed there is a unique way to express any vector as a linear combination of basis vectors. We proceed to the problems:

(a) There is exactly one solution. Observe that a solution $x, z \in \mathbf{R}$ to the equation is the same thing as a solution $x, z \in \mathbf{R}$ to the following equation of vectors:

$$x\begin{pmatrix}1\\2\end{pmatrix} + z\begin{pmatrix}17\\1\end{pmatrix} = (3, 0).$$

Now we will apply our general observation. We claim $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}17\\1\end{pmatrix}\right\}$ is a basis of $\mathbf{R}^2$. Indeed, by the two-out-of-three criteria we just need to show that they are linearly independent: but this is clear because neither vector is a scalar multiple of the other. So indeed there is a unique such solution $x, y \in \mathbf{R}$.

(b) Again, there is exactly one solution. Again, a solution $x, y, z \in \mathbf{R}$ to the equation is the same as a solution $x, y, z \in \mathbf{R}$ to the following equation of vectors:

$$x \begin{pmatrix} 5 \\ 19 \\ -113 \end{pmatrix} + y \begin{pmatrix} -7 \\ 12 \\ 1 \end{pmatrix} + z \begin{pmatrix} 17 \\ -9 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So again we will apply our general observation. We claim

$$S := \left\{ \begin{pmatrix} 5 \\ 19 \\ -113 \end{pmatrix}, \begin{pmatrix} -7 \\ 12 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ -9 \\ -1 \end{pmatrix} \right\}$$

is a basis of $\mathbf{R}^3$. To show this, by the two-out-of-three criterion, we can show it spans $\mathbf{R}^3$; in particular we can express $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as linear combinations of elements of S as follows:

$$-\frac{3}{16108} \begin{pmatrix} 5 \\ 19 \\ -113 \end{pmatrix} + \frac{259}{4027} \begin{pmatrix} -7 \\ 12 \\ 1 \end{pmatrix} + \frac{1375}{16108} \begin{pmatrix} 17 \\ -9 \\ -1 \end{pmatrix} = \vec{e}_1$$

$$\frac{5}{8054} \begin{pmatrix} 5 \\ 19 \\ -113 \end{pmatrix} + \frac{479}{4027} \begin{pmatrix} -7 \\ 12 \\ 1 \end{pmatrix} + \frac{393}{8054} \begin{pmatrix} 17 \\ -9 \\ -1 \end{pmatrix} = \vec{e}_2$$

$$-\frac{141}{16108} \begin{pmatrix} 5 \\ 19 \\ -113 \end{pmatrix} + \frac{92}{4027} \begin{pmatrix} -7 \\ 12 \\ 1 \end{pmatrix} + \frac{193}{16108} \begin{pmatrix} 17 \\ -9 \\ -1 \end{pmatrix} = \vec{e}_3$$

(c) There are infinitely many solutions. A solution $u, v, w, x, y, z \in \mathbf{R}$ to the equation is the same thing as a solution $u, v, w, x, y, z \in \mathbf{R}$ to the following equation of vectors:

$$u\vec{r}_1 + v\vec{r}_2 + w\vec{r}_3 + x\vec{r}_4 + y\vec{r}_5 + z\vec{r}_6 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

where

$$\vec{r}_1 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{r}_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{r}_3 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{r}_4 := \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \qquad \vec{r}_5 := \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{r}_6 := \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

First note that $u = 2, v = -1, w = 0, x = 1, y = 0, z = 0$ is one solution. Next, note that $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4, \vec{r}_5, \vec{r}_6$ must be linearly dependent in $\mathbf{R}^4$, just because the maximal size of set of linearly independent vectors in $\mathbf{R}^4$ is the dimension of the space, namely, 4. But that means we can find $\alpha_1, \ldots, \alpha_6 \in \mathbf{R}$ such that

$$\alpha_1\vec{r}_1 + \alpha_2\vec{r}_2 + \alpha_3\vec{r}_3 + \alpha_4\vec{r}_4 + \alpha_5\vec{r}_5 + \alpha_6\vec{r}_6 = \vec{0}$$

and so that not α_i all zero. But then

$$(2+t\alpha_1)\vec{r}_1 + (-1+t\alpha_2)\vec{r}_2 + t\alpha_3\vec{r}_3 + (1+t\alpha_4)\vec{r}_4 + t\alpha_5\vec{r}_5 + t\alpha_6\vec{r}_6 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

for all $t \in \mathbf{R}$, and these are all different because $\alpha_i \neq 0$ for some i, so indeed we have infinitely many solutions.

(d) There is exactly one solution. Observe that a solution$u, v, w, x, y, z \in \mathbf{R}$ to the equation is the same thing as a solution $u, v, w, x, y, z \in \mathbf{R}$ to the following equation of vectors:

$$u\vec{r}_6 + v\vec{r}_5 + w\vec{r}_4 + x\vec{r}_3 + y\vec{r}_2 + z\vec{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\vec{r}_i = -3\vec{e}_i + \sum_{j=1}^{6} \vec{e}_j$. Here the $\vec{e}_j$ are the standard basis vectors of $\mathbf{R}^6$. As in (1) and (2) above, we will apply our general observation.

To that end, we claim that $S := \{\vec{r}_1, \ldots, \vec{r}_6\}$ is a basis of $\mathbf{R}^6$. To show this, by the two-out-of-three criterion, we can show it spans $\mathbf{R}^6$; in particular we can express $\vec{e}_1, \ldots, \vec{e}_6$ as linear combinations of elements of S as follows:

$$\begin{aligned}
-\frac{2}{9}\vec{r}_1 + \frac{1}{9}\vec{r}_2 + \frac{1}{9}\vec{r}_3 + \frac{1}{9}\vec{r}_4 + \frac{1}{9}\vec{r}_5 + \frac{1}{9}\vec{r}_6 &= \vec{e}_1 \\
\frac{1}{9}\vec{r}_1 - \frac{2}{9}\vec{r}_2 + \frac{1}{9}\vec{r}_3 + \frac{1}{9}\vec{r}_4 + \frac{1}{9}\vec{r}_5 + \frac{1}{9}\vec{r}_6 &= \vec{e}_2 \\
\frac{1}{9}\vec{r}_1 + \frac{1}{9}\vec{r}_2 - \frac{2}{9}\vec{r}_3 + \frac{1}{9}\vec{r}_4 + \frac{1}{9}\vec{r}_5 + \frac{1}{9}\vec{r}_6 &= \vec{e}_3 \\
\frac{1}{9}\vec{r}_1 + \frac{1}{9}\vec{r}_2 + \frac{1}{9}\vec{r}_3 - \frac{2}{9}\vec{r}_4 + \frac{1}{9}\vec{r}_5 + \frac{1}{9}\vec{r}_6 &= \vec{e}_4 \\
\frac{1}{9}\vec{r}_1 + \frac{1}{9}\vec{r}_2 + \frac{1}{9}\vec{r}_3 + \frac{1}{9}\vec{r}_4 - \frac{2}{9}\vec{r}_5 + \frac{1}{9}\vec{r}_6 &= \vec{e}_5 \\
\frac{1}{9}\vec{r}_1 + \frac{1}{9}\vec{r}_2 + \frac{1}{9}\vec{r}_3 + \frac{1}{9}\vec{r}_4 + \frac{1}{9}\vec{r}_5 - \frac{2}{9}\vec{r}_6 &= \vec{e}_6.
\end{aligned}$$

Problem 4 *What's the angle between the following vectors? Compute the projection $\pi_{\vec{a}}(\vec{b})$ in each case.*

(a) $\vec{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ *and* $\vec{b} = \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

(b) $\vec{a} = \begin{pmatrix} 4 \\ -4 \\ 7 \end{pmatrix}$ *and* $\vec{b} = \begin{pmatrix} -1 \\ 4 \\ -8 \end{pmatrix}$

(c) $\vec{a} = \begin{pmatrix} 169 \\ -520 \\ -561 \\ 425 \end{pmatrix}$ *and* $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

(d) $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ *and* $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Solution: In all cases below we use θ to denote the angle between $\vec{a}$ and $\vec{b}$:

(a) We know $\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{26}{\sqrt{9}\sqrt{169}} = \frac{2}{3}$. Let us use arccos to denote the unique bijective function from $[-1,1]$ to $[0,\pi]$ that satisfies $\arccos(\cos(\theta)) = \theta$ for all $\theta \in [0,\pi]$. Thus $\theta = \arccos(\frac{2}{3}) \approx 48.19°$. Then the projection $\pi_{\vec{a}}(\vec{b})$ is $\pi_{\vec{a}}(\vec{b}) = \frac{|\vec{b}|}{|\vec{a}|}\cos(\theta)\vec{a} = \frac{26}{9}\vec{a} = \begin{pmatrix} \frac{52}{9} \\ \frac{52}{9} \\ \frac{26}{9} \end{pmatrix}$.

(b) We have $\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-76}{\sqrt{81}\sqrt{81}} = \frac{-76}{81}$. Thus $\theta = \arccos(\frac{-76}{81}) \approx 159.8°$. And the projection is $\pi_{\vec{a}}(\vec{b}) = \frac{|\vec{b}|}{|\vec{a}|}\cos(\theta)\vec{a} = -\frac{76}{81}\vec{a} = \begin{pmatrix} \frac{304}{81} \\ \frac{-304}{81} \\ \frac{532}{81} \end{pmatrix}$.

(c) We have $\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{297}{\sqrt{794307}\sqrt{4}} = \frac{297}{2\sqrt{794307}}$. Thus $\theta = \arccos(\frac{297}{2\sqrt{794307}}) \approx 80.4°$. And the projection is $\pi_{\vec{a}}(\vec{b}) = \frac{|\vec{b}|}{|\vec{a}|}\cos(\theta)\vec{a} = \frac{297}{794307}\vec{a} = \begin{pmatrix} \frac{16731}{264769} \\ \frac{-51480}{264769} \\ \frac{-55539}{264769} \\ \frac{42075}{264769} \end{pmatrix}$.

(d) We have $\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{\sqrt{4}\sqrt{4}} = \frac{1}{2}$. Thus $\theta = \arccos(\frac{1}{2}) = 60°$ (or $\frac{\pi}{3}$ radians). And the projection is $\pi_{\vec{a}}(\vec{b}) = \frac{|\vec{b}|}{|\vec{a}|}\cos(\theta)\vec{a} = \frac{1}{2}\vec{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$.

Problem 5 *What's the length of the vector*

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ 23 \\ 24 \end{pmatrix} \in \mathbf{R}^{25}?$$

Solution: It can easily be proved by induction that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

for all $n = 0, 1, 2, \ldots$. Thus the length of this vector is $\sqrt{0^2 + 1^2 + \cdots + 24^2} = \sqrt{\frac{24(25)(49)}{6}} = 70$.

Problem 6 *Show that any unit vector $\hat{u} \in \mathbf{R}^{n+1}$ can be written as*

$$\hat{u} = \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ \vdots \\ \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\cos(\theta) \\ \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\sin(\theta) \end{pmatrix}$$

with $\phi_1, \phi_2, \ldots, \phi_{n-1} \in [0, \pi]$ and $\theta \in [0, 2\pi)$. Draw a picture for $n = 1$ and $n = 2$ to illustrate.

Solution: Let $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{pmatrix}$ be a unit vector in $\mathbf{R}^{n+1}$. We proceed to define $\phi_1, \phi_2, \ldots, \phi_{n-1} \in [0, \pi]$ and $\theta \in [0, 2\pi)$ so that $\hat{u}$ is as in the statement of the problem. First let us define the ϕ_i. We will do so recursively. Suppose that we have already found $\phi_1, \ldots, \phi_{i-1} \in [0, \pi]$ so that

$$\begin{aligned} u_1 &= \cos(\phi_1) \qquad (1) \\ u_2 &= \sin(\phi_1)\cos(\phi_2) \\ &\vdots \\ u_{i-1} &= \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-2})\cos(\phi_{i-1}) \end{aligned}$$

We want to find a $\phi_i \in [0, \pi]$ so that

$$u_i = \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})\cos(\phi_i) \qquad (2)$$

To that end, we claim that for $1 \leq k \leq i - 1$ we have

$$\sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_k) = 1 - u_1^2 - u_2^2 - \cdots - u_k^2. \qquad (3)$$

The case $k = 1$ of (3) follows from the assumption in (1) that $u_1 = \cos(\phi_1)$. So suppose $k > 1$ and the claim holds for $k - 1$. Then

$$\begin{aligned} \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_k) &= \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{k-1})(1 - \cos^2(\phi_k)) \\ &= \sin^2(\phi_1)\cdots\sin^2(\phi_{k-1}) - \sin^2(\phi_1)\cdots\sin^2(\phi_{k-1})\cos^2(\phi_k) \\ &= 1 - u_1 - u_2 - \cdots - u_{k-1} - u_k \end{aligned}$$

where in the last line we use our inductive hypothesis and the assumption in (1) that $u_k = \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{k-1})\cos^2(\phi_k)$. So indeed (3) holds. Now we proceed to define ϕ_i to satisfy (2). Note that $\hat{u}$ being a unit vector is equivalent to $u_1^2 + u_2^2 + \cdots + u_{n+1}^2 = 1$. So in particular we have

$$0 \leq 1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2 \leq 1.$$

First suppose that $1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2 = 0$. Then note that $u_i = 0$ because otherwise $u_1^2 + u_2^2 + \cdot + u_{n+1}^2 > 1$. Thus in this case we can choose *any* $\phi_i \in [0, \pi]$ and (2) will be satisfied, since by (3) we have

$$\sin^2(\phi_1)\cdots\sin^2(\phi_i) = 1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2 = 0$$

which implies

$$\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_i) = 0.$$

So now let us suppose that $0 < 1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2 \leq 1$. Then note that

$$0 \leq u_i^2 \leq 1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2,$$

again by using the fact that $\hat{u}$ is a unit vector. Diving through we get

$$0 \leq \frac{u_i^2}{1 - u_1^2 - u_2^2 - \cdots - u_{i-1}^2} \leq 1,$$

and then taking square roots and using (3) we have

$$0 \leq \left| \frac{u_i}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})} \right| \leq 1.$$

So in this case we can define $\phi_i := \arccos\left(\frac{u_i}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})}\right) \in [0, \pi]$ and we will satisfy (2).

We have now successfully defined $\phi_1, \ldots, \phi_{n-1} \in [0, \pi]$ so that

$$\begin{aligned} u_1 &= \cos(\phi_1) \\ u_2 &= \sin(\phi_1)\cos(\phi_2) \\ &\vdots \\ u_{n-1} &= \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1}) \end{aligned}$$

Moreover, the same argument used to establish (3) still applies to $i := n - 1$ and so we have

$$\sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{n-1}) = 1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2. \tag{4}$$

We want to find $\theta \in [0, 2\pi)$ so that

$$\begin{aligned} u_n &= \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\cos(\theta) \\ u_{n+1} &= \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\sin(\theta) \end{aligned} \tag{5}$$

As before we have $0 \leq 1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2 \leq 1$. Suppose $1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2 = 0$. Then $u_n = u_{n+1} = 0$ again because $\hat{u}$ is a unit vector. In this case we can choose *any* $\theta \in [0, 2\pi)$ and we will satisfy (5) because by (4) we have

$$\sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{n-1}) = 1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2 = 0$$

which implies

$$\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1}) = 0.$$

So now suppose $0 < 1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2 \leq 1$. Then as before

$$0 \leq \left| \frac{u_n}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})} \right| \leq 1.$$

Let us define $\theta \in [0, 2\pi)$ by

$$\theta := \begin{cases} \arccos\left(\frac{u_n}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})}\right) & \text{if } \frac{u_{n+1}}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})} \geq 0, \\ 2\pi - \arccos\left(\frac{u_n}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})}\right) & \text{otherwise.} \end{cases}$$

Why does this definition satisfy (5)? Well, it clearly satisfies the first equation in (5) because

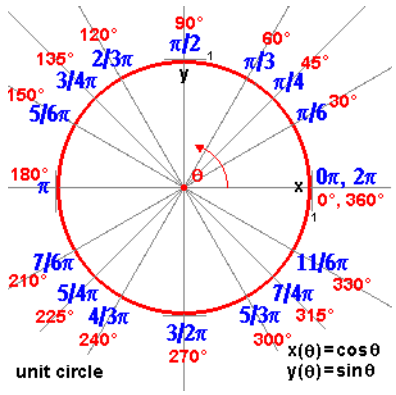
$$\begin{aligned} \cos\left(2\pi - \arccos\left(\frac{u_i}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})}\right)\right) &= \cos\left(\arccos\left(\frac{u_i}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})}\right)\right) \\ &= \frac{u_i}{\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{i-1})} \end{aligned}$$

because cos is even and has period 2π. And as to the second equation in (5) we can check that

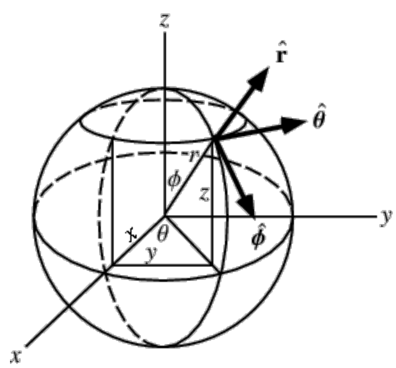
$$\begin{aligned} \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{n-1})\sin^2(\theta) &= \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{n-1})(1 - \cos^2(\theta)) \\ &= \sin^2(\phi_1)\cdots\sin^2(\phi_{n-1}) - \sin^2(\phi_1)\cdots\sin^2(\phi_{n-1})\cos^2(\theta) \\ &= 1 - u_1^2 - u_2^2 - \cdots - u_{n-1}^2 - u_n^2 \\ &= u_{n+1}^2 \end{aligned}$$

because $1 = \sum_{i=1}^{n+1} u_i^2$. So $\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\sin(\theta) = \pm u_{n+1}$, and the two cases in our definition θ deal with this choice of sign.

The case $n = 1$ is the well-known polar coordinates:



The case $n = 2$ is spherical coordinates:



Problem 7 *Suppose $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k \in \mathbf{R}^n$ is a collection of vectors such that*

$$\hat{u}_i \cdot \hat{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Show that $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k$ are linearly independent.

Solution: Let $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k \in \mathbf{R}^n$ satisfy the above property. Suppose that

$$\alpha_1 \hat{u}_1 + \alpha_2 \hat{u}_2 + \cdots + \alpha_k \hat{u}_k = \vec{0}$$

for $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$. By taking the dot product of the above equation with $\hat{u}_i$, we see that $\alpha_i = 0$. Thus for all i, $\alpha_i = 0$, which means our solution must've been trivial. So indeed the vectors are linearly independent.