# PROBLEM SET I

DUE 16 FEBRUARY 2016

(1) Are the following collections of vectors in $\mathbf{R}^3$ linearly independent? Why or why not?

(a) $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

(b) $\left\{\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right\}$

(c) $\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix}\right\}$

(d) $\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.00001 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix}\right\}$

(e) $\left\{\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix}\right\}$

(f) $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$

(g) $\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$

(2) Write, if possible, the vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3 \in \mathbf{R}^3$ as a linear combination of the following collections of vectors. If it is not possible, explain why not.

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.00001 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix} \right\}$

(f) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(g) $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

(3) How many solutions does each of the following systems of linear equations have? (Answer *without* solving them, if you can!)

(a)

$$\begin{aligned} x + 17z &= 3 \\ 2x + z &= 0 \end{aligned}$$

(b)

$$\begin{aligned} 5x - 7y + 17z &= 2 \\ 19x + 12y - 9z &= 88 \\ -113x + y - z &= -1 \end{aligned}$$

(c)

$$\begin{aligned} x + y + 2z &= 1 \\ w + x + 2y &= 1 \\ v + w + 2x &= 1 \\ u + v + 2w &= 1 \end{aligned}$$

(d)

$$\begin{aligned} u + v + w + x + y - 2z &= 0 \\ u + v + w + x - 2y + z &= 0 \\ u + v + w - 2x + y + z &= 0 \\ u + v - 2w + x + y + z &= 0 \\ u - 2v + w + x + y + z &= 0 \\ -2u + v + w + x + y + z &= 0 \end{aligned}$$

(4) What's the angle between the following vectors? Compute the projection $\pi_{\vec{a}}(\vec{b})$ in each case.

(a) $\vec{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

(b) $\vec{a} = \begin{pmatrix} 4 \\ -4 \\ 7 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 4 \\ -8 \end{pmatrix}$

(c) $\vec{a} = \begin{pmatrix} 169 \\ -520 \\ -561 \\ 425 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

(d) $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(5) What's the length of the vector

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ 23 \\ 24 \end{pmatrix} \in \mathbf{R}^{25}?$$

(6) Show that any unit vector $\widehat{u} \in \mathbf{R}^{n+1}$ can be written as

$$\widehat{u} = \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ \vdots \\ \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\cos(\theta) \\ \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\sin(\theta) \end{pmatrix}$$

with $\phi_1, \phi_2, \ldots, \phi_{n-1} \in [0, \pi]$ and $\theta \in [0, 2\pi)$. Draw a picture for $n = 1$ and $n = 2$ to illustrate.

(7) Suppose $\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_k \in \mathbf{R}^n$ a collection of vectors such that

$$\widehat{u}_i \cdot \widehat{u}_j = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j. \end{cases}$$

Show that $\{\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_k\}$ are linearly independent.