18.06.29: Complex matrices

Lecturer: Barwick

If it can be used again,
it is not wisdom but theory.
— James Richardson

Proposition. *Any complex vector subspace $W \subset \mathbf{C}^n$ of complex dimension k has an underlying real vector space of dimension $2k$.*

To see why, take a $\mathbf{C}$-basis $\{w_1, \ldots, w_k\}$ of W. Now $\{w_1, iw_1, \ldots, w_k, iw_k\}$ is an $\mathbf{R}$-basis of W.

In the other direction, a real vector subspace $V \subseteq \mathbf{R}^n$ generates a complex vector subspace $V_{\mathbf{C}} \subseteq \mathbf{C}^n$, called the *complexification*; this is the set of all $\mathbf{C}$-linear combinations of elements of V:

$$V_{\mathbf{C}} \coloneqq \left\{ w \in \mathbf{C}^n \,\middle|\, w = \sum_{i=1}^{k} \alpha_i v_i, \text{ for some } \alpha_1, \ldots, \alpha_k \in \mathbf{C},\ v_1, \ldots, v_k \in V \right\}.$$

Note that not all complex vector subspaces of $\mathbf{C}^n$ are themselves complexifications; the complex vector subspace $W \subset \mathbf{C}^2$ spanned by $\begin{pmatrix} i \\ 1 \end{pmatrix}$ provides a counterexample. (A complex vector space is a complexification if and only if it has a $\mathbf{C}$-basis consisting of real vectors.)

Now, most importantly, we may speak of *complex matrices* (i.e., matrices with complex entries).

All the algebra we've done with matrices over $\mathbf{R}$ *works perfectly for matrices over* $\mathbf{C}$*, without change.*

However, the freedom to contemplate complex matrices offers us new horizons when it comes to questions about eigenspaces and diagonalization. Let's contemplate the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial $p_A(t) = t^2 + 1$ doesn't have any real roots, so there's no hope of diagonalizing A over $\mathbf{R}$.

Over $\mathbf{C}$, however, we find eigenvalues $i, -i$. Let's try to diagonalize A.

Let's begin with $L_i = \ker(iI - A) = \ker\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$. It's dimension 1, and it's spanned by the vector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

And $L_{-i} = \ker\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$ is dimension 1 and spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Note that neither L_i nor L_{-i} is a complexification. However, we do have a basis $\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$ of $\mathbf{C}^2$ consisting of eigenvectors of A, and writing T_A in terms of this basis gives us the matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

So A is not diagonalizable over $\mathbf{R}$, but it is diagonalizable over $\mathbf{C}$.

More generally, if we're looking at a real matrix of the form

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

then $M = M_z$ for $z = a + bi$, and on the problem set, you'll show that

$$p_M(t) = t^2 - (z + \overline{z})t + z\overline{z}.$$

The roots of this polynomial are z and $\overline{z}$.

So, very pleasantly, M is diagonalizable over **C**, and it's similar to the matrix

$$M = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}.$$

This game of going back and forth between z and M_z is helpful in other ways. For example, let's take a 2×3 complex matrix

$$A = \begin{pmatrix} 1 & 2i & 2+3i \\ 1-4i & 5i & 1-i \end{pmatrix}.$$

We can replace each complex entry z with the 2×2 matrix M_z that corresponds to it, giving us a 4×6 real matrix M_A …

$$M_A = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & -2 & 2 & -3 \\ 0 & 1 & 2 & 0 & 3 & 2 \\ \hline 1 & 4 & 0 & -5 & 1 & 1 \\ -4 & 1 & 5 & 0 & -1 & 1 \end{array}\right).$$

How is that helpful? Well, if we think of $T_A : \mathbf{C}^3 \longrightarrow \mathbf{C}^2$ given by multiplication by A, we should be able to regard that as a linear map $\mathbf{R}^6 \longrightarrow \mathbf{R}^4$ given by a 4×6 matrix. M_A is precisely that matrix!

In particular, think about the transpose of M_A. What complex matrix does it correpond to?

Our last midterm is Friday. (sniff!)

- I know you're sad, but try to work through the hurt.
- Five questions, as usual.
- It covers everything up to this page of the lectures.
- I'm aiming for a mean of around 90 again. I missed last time, but I suspect that had more to do with the shittiness of that particular week than with your ability to do the math.

Back to our transpose:

$$M_A^\intercal = \left(\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & 4 & 1 \\ \hline 0 & 2 & 0 & 5 \\ -2 & 0 & -5 & 0 \\ \hline 2 & 3 & 1 & -1 \\ -3 & 2 & 1 & 1 \end{array}\right).$$

and let's convert it back to a complex matrix …

$$A^* = \begin{pmatrix} 1 & 1+4i \\ -2i & -5i \\ 2-3i & 1+i \end{pmatrix}.$$

This is the *conjugate transpose* of A, so that

$$A^* = \overline{(A^\intercal)} = \left(\overline{A}\right)^\intercal .$$

This clearly works in general, and we therefore find that $M_A^\intercal = M_{A^*}$.

A *real matrix* A is said to be *symmetric* if $A = A^{\intercal}$.

A *complex matrix* B is said to be *Hermitian* if M_B is symmetric – or, equivalently, if $B = B^*$.

(Note that a Hermitian matrix with real entries must be symmetric.)

We need to think about this a bit more carefully. For that, let's contemplate the correct version of the dot product in $\mathbf{C}^n$, and develop some notation.

For $v, w \in \mathbf{C}^n$, write

$$\langle v|w\rangle \coloneqq v^* w.$$

This is a complex number, called the *inner product* of two complex vectors; it extends the usual dot product, but notices that the linearity in the first coordinate is *twisted*:

$$\langle \alpha v|w\rangle = \overline{\alpha}\langle v|w\rangle.$$

With this, one can repeat the usual definition of orthogonality with no problem.

Lemma. *An $n \times n$ complex matrix B is Hermitian if and only if, for any $v, w \in \mathbf{C}^n$,*

$$\langle Av|w\rangle = \langle v|Aw\rangle.$$

Theorem (Spectral theorem; last big result of the semester). *Suppose B a Hermitian matrix. Then*

(1) The eigenvalues of B are real.

(2) There is an orthogonal basis of eigenvectors for B; in particular, B is diagonalizable over **C** *(and even over* **R** *if B has real entries).*