18.06.28: Complex vector spaces

Lecturer: Barwick

I worked so hard to understand it that it must be true.

— James Richardson

From last time …

I was alluding to a way to make complex multiplication easier to understand. The idea is this: for any $z = a + bi \in \mathbf{C}$, you may consider the matrix

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

On the newest problem set, you'll show that addition of complex numbers is addition of these matrices, multiplication of complex numbers is multiplication of these matrices (!), and one more thing …

Complex conjugation is the map $z \rightsquigarrow \overline{z}$ that carries $z = a + bi$ to $\overline{z} = a - bi$. You'll see that $M_{\overline{z}} = M_z^{\intercal}$.

Complex conjugation can be used to extract the real and complex parts of your complex number:

$$\begin{aligned} 2a &= z + \overline{z}; \\ 2bi &= z - \overline{z}. \end{aligned}$$

Complex conjugation also gives you the length of the vector $\vec{v} \in \mathbf{R}^2$ corresponding to $z \in \mathbf{C}$:

$$\|\vec{v}\|^2 = z\overline{z}.$$

So

$$z = \sqrt{z\overline{z}}\,\exp(i\theta)$$

for some (and hence infinitely many) $\theta \in \mathbf{R}$.

If $z \neq 0$, there is a unique such $\theta \in [0, 2\pi)$; this is sometimes called the *argument* of z, but it's annoying to write down a good formula. It's better to think of it as an element of $\mathbf{R}/2\pi\mathbf{Z}$.

One last general thing about the complex numbers, just because it's so important.

Theorem ("Fundamental theorem of algebra"). *For any polynomial*

$$f(z) = \sum_{i=0}^{n} \alpha_i z^i$$

with complex coefficients α_i *such that* $\alpha_n \neq 0$, *there exist complex numbers* $w_1, \ldots, w_n$ *such that*

$$f(z) = \alpha_n (z - w_1)(z - w_2) \cdots (z - w_n).$$

This is actually a theorem of *topology*, not algebra, but there you go.

Here's a cool example: $f(z) = z^n - 1$. Let's find the roots!

The set $\mathbf{C}^n$ is the set of column vectors

$$v = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

with $z_i \in \mathbf{C}$. One can add such vectors componentwise, and one can multiply any such vector with a *complex* scalar.

So this is the fundamental example of a *complex vector space*.

Note that $\mathbf{R}^n \subset \mathbf{C}^n$. Now if $v \in \mathbf{C}^n$, then $v = \overline{v}$ if and only if $v \in \mathbf{R}^n$.

More generally, a *complex vector subspace* $V \subseteq \mathbf{C}^n$ is a subset such that:

(1) for any $v, w \in V$, one has $v + w \in V$;

(2) for any $v \in V$ and any $z \in \mathbf{C}$, one has $zv \in V$.

Vectors $v_1, \ldots, v_k$ *span* a vector subspace $V \subseteq \mathbf{C}^n$ *over* **C** if and only if every vector $w \in V$ can be written as a **C**-*linear combination* of the v_i, i.e.,

$$w = \sum_{i=1}^{k} z_i v_i,$$

where each $z_i \in \mathbf{C}$.

Similarly, the vectors $v_1, \ldots, v_k$ are *linearly independent over* $\mathbf{C}$ if and only if any vanishing $\mathbf{C}$-linear combination

$$\sum_{i=1}^{k} z_i v_i = 0$$

is a trivial $\mathbf{C}$-linear combination, so that $z_1 = \cdots = z_k = 0$.

A $\mathbf{C}$-*basis* of V is thus a collection of vectors of V that is linearly independent over $\mathbf{C}$ and spans V over $\mathbf{C}$.

Let's do an example to appreciate the distinction. Let's think of $\mathbf{C}^2$, and let's think of the complex line

$$L = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbf{C}^2 \;\middle|\; 3z - 2w = 0 \right\} \subset \mathbf{C}^2.$$

Now $\mathbf{C}^2 \cong \mathbf{R}^4$, so that complex line *is* a real plane:

$$L = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{pmatrix} \in \mathbf{R}^4 \;\middle|\; \begin{aligned} 3z_1 - 2w_1 &= 0 \\ 3z_2 - 2w_2 &= 0 \end{aligned} \right\} \subset \mathbf{R}^4.$$

The single vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ forms a $\mathbf{C}$-basis of L.

Another legit $\mathbf{C}$-basis would be the single vector $\begin{pmatrix} 2i \\ 3i \end{pmatrix}$.

The vectors

$$\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} \right\}$$

forms an $\mathbf{R}$-basis of L over $\mathbf{R}$.

Another example: consider the complex vector subspace $W \subset \mathbf{C}^2$ spanned by $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

Here's an important sentence to parse correctly: *W does not have a* $\mathbf{C}$*-basis consisting of real vectors.*

A real basis for $W \subset \mathbf{R}^4$ consists of $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Proposition. *Any complex vector subspace $W \subset \mathbf{C}^n$ of complex dimension k has an underlying real vector space of dimension $2k$.*

To see why, take a $\mathbf{C}$-basis $\{w_1, \ldots, w_k\}$ of W. Now $\{w_1, iw_1, \ldots, w_k, iw_k\}$ is an $\mathbf{R}$-basis of W.

In the other direction, a real vector subspace $V \subseteq \mathbf{R}^n$ generates a complex vector subspace $V_{\mathbf{C}} \subseteq \mathbf{C}^n$, called the *complexification*; this is the set of all $\mathbf{C}$-linear combinations of elements of V:

$$V_{\mathbf{C}} \coloneqq \left\{ w \in \mathbf{C}^n \;\middle|\; w = \sum_{i=1}^{k} \alpha_i v_i, \text{ for some } \alpha_1, \ldots, \alpha_k \in \mathbf{C},\ v_1, \ldots, v_k \in V \right\}.$$

Note that not all complex vector subspaces of $\mathbf{C}^n$ are themselves complexifications; the complex vector subspace $W \subset \mathbf{C}^2$ spanned by $\begin{pmatrix} i \\ 1 \end{pmatrix}$ provides a counterexample. (A complex vector space is a complexification if and only if it has a $\mathbf{C}$-basis consisting of real vectors.)

Now, most importantly, we may speak of *complex matrices* (i.e., matrices with complex entries).

All the algebra we've done with matrices over $\mathbf{R}$ *works perfectly for matrices over* $\mathbf{C}$*, without change.*

However, the freedom to contemplate complex matrices offers us new horizons when it comes to questions about eigenspaces and diagonalization. Let's contemplate the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial $p_A(t) = t^2 + 1$ doesn't have any real roots, so there's no hope of diagonalizing A over $\mathbf{R}$.

Over $\mathbf{C}$, however, we find eigenvalues $i, -i$. Let's try to diagonalize A.

Let's begin with $L_i = \ker(iI - A) = \ker\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$. It's dimension 1, and it's spanned by the vector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

And $L_{-i} = \ker\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$ is dimension 1 and spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Note that neither L_i nor L_{-i} is a complexification. However, we do have a basis $\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$ of $\mathbf{C}^2$ consisting of eigenvectors of A, and writing T_A in terms of this basis gives us the matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

So A is not diagonalizable over $\mathbf{R}$, but it is diagonalizable over $\mathbf{C}$.

There's one more new thing you can do with a complex matrices that doesn't quite work for real matrices: you can conjugate their entries. Of particular import is the *conjugate transpose*:

$$A^* := \left(\overline{A}\right)^{\intercal} = \overline{(A^{\intercal})}.$$

We'll understand the significance of this operation next time.