18.06.18: More fun with spacetime

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Last time, we introduced the following model for our universe: take $\mathbf{R}^4$ (with standard basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4)$). All the geometry comes from the matrix

$$H = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we write $\eta(\vec{v}, \vec{w}) \coloneqq \underset{\rightarrow}{v} H \vec{w}$.

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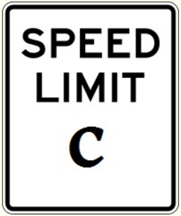

For any vector $\vec{v} \in \mathbf{R}^4$, write $s^2(\vec{v}) = \eta(\vec{v}, \vec{v}) \in \mathbf{R}$. If $s^2(\vec{v}) > 0$, we say that $\vec{v}$ is *spacelike*; if $s^2(\vec{v}) < 0$, we say that $\vec{v}$ is *timelike*; if $s^2(\vec{v}) = 0$, we say that $\vec{v}$ is *lightlike*. This gave us our *light cone*.

A 4×4 matrix M such that

$$H = M^T HM.$$

is called a *Lorentz transformation*. This is defined precisely so that

$$\eta(M\vec{v}, M\vec{w}) = \eta(\vec{v}, \vec{w}).$$

Any physical laws we discover should be *invariant* under Lorentz transformations; this includes the relativistic laws of mechanics, Maxwell's field equations, and the Dirac equation.

Orthogonal matrices are matrices R such that $R^T R = I$. In effect, they preserve all the geometry given by the dot product.

In 2 dimensions, we can write them all down; they all look like

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

In 3 dimensions, orthogonal matrices are products of matrices that rotate about an axis and reflections in planes. (It's quite tricky to write all of them down in terms of sines and cosines!)

Any 3×3 *orthogonal* matrix R (i.e., a matrix R such that), the block matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

is a Lorentz transformation.

So if we choose a basis $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ for $\mathbf{R}^3$ such that $x_i \cdot x_j = \delta_{ij}$, then we get a Lorentz basis $(\hat{e}_1, \vec{x}_1, \vec{x}_2, \vec{x}_3)$.

Any laws of physics we derive relative to $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4)$ will work relative to $(\hat{e}_1, \vec{x}_1, \vec{x}_2, \vec{x}_3)$.

There is a different sort of Lorentz basis as well. Consider an observer at the origin, moving in the positive x direction with speed u (recall $c = 1$). This observer will agree with a stationary observer at the origin about the direction of the x, y, and z axes, and it will agree with the stationary observer's measurement of length in the y and z directions; however, this observer will see the x and t directions very differently…

Write $\phi \coloneqq \tanh^{-1}(u)$. The matrix

$$\Lambda_\phi = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) & 0 & 0 \\ \sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-u^2}} & \frac{u}{\sqrt{1-u^2}} & 0 & 0 \\ \frac{u}{\sqrt{1-u^2}} & \frac{1}{\sqrt{1-u^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a Lorentz transformation. This is the *Lorentz boost* in the positive x-direction at speed u. Physically, this means that an observer moving in the positive x direction with speed u will see a vector v in spacetime as $\Lambda_\phi v$.

This accounts for phenomena such as *time dilation* and *Lorentz contraction*. (How?)

Lorentz transformations can be divided into four sorts, based on whether $\det(\Lambda)$ is +1 or −1, and whether the upper left entry Λ_{11} is positive or negative:

$\det(\Lambda)$	$\mathrm{sgn}(\Lambda_{11})$	transformation
+1	+	proper, isochronous
-1	+	space inverting, isochronous
-1	-	space inverting, time reversing
+1	-	proper, time reversing

0.1 Proposition. *Any proper, isochronous Lorentz transformation is the product of a spatial rotation and a boost.*