18.06.14: ‘Rank-nullity’

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Monday 7 March 2016

Theorem (Rank-Nullity Theorem). *If A is an $m \times n$ matrix, then*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$$

Suppose A were in reduced row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Then, we have a few columns (1, 3, 4, 5 in this case) that are distinct standard basis vectors, and the other columns can be written as a linear combination of these. So the rank here is 4.

When we have a vector $\vec{x}$ with $A\vec{x} = \vec{0}$, we can write x_1, x_3, x_4, and x_5 in terms of the other variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -7 \\ -6 \\ -7 \\ 1 \end{pmatrix}.$$

And as we know, $4 + 2 = 6$.

This pattern works in general. A matrix A is in rref iff it looks like this:

$$\begin{pmatrix} \vec{A}^1 & \cdots & \vec{A}^{k_1-1} & \hat{e}_1 & \vec{A}^{1+k_1} & \cdots & \vec{A}^{k_2-1} & \hat{e}_2 & \cdots & \hat{e}_r & \vec{A}^{1+k_r} & \cdots & \vec{A}^n \end{pmatrix}.$$

where the column vectors $\vec{A}^{1+k_i}, \ldots, \vec{A}_{k_{i+1}-1}$ all lie in the span of $\hat{e}_1, \ldots, \hat{e}_i$, but not in the span of $\hat{e}_1, \ldots, \hat{e}_{i-1}$.

The image is thus the span of the of distinct $\hat{e}_i$'s that appear:

$$r = \dim(\mathrm{im}(A)).$$

When you compute the kernel, on the other hand, you express $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ in terms of the other x_i's. That is, you write any $\vec{x} \in \ker(A)$ as a linear combination of vectors

$$\vec{v}_1, \ldots, \vec{v}_{k_1-1}, \vec{v}_{1+k_1}, \ldots, \vec{v}_{k_2-1}, \ldots, \vec{v}_{1+k_r}, \ldots, \vec{v}_n,$$

where each $\vec{v}_j$ has a 1 in the jth spot, *something* in spots $k_1, k_2, \ldots, k_r$, and a 0 in every other spot.

In particular, these vectors must be linearly independent, so they form a basis of the kernel! And since there are $n - r$ of them, we have

$$\dim(\ker(A)) = n - r,$$

as desired.

This proves the Rank-Nullity Theorem when A is in rref!

But what if A isn't in rref? What then?

Well, we know we can perform row operations to get A into rref. (This is the magic of Gaussian elimination.)

$$\textit{row operations}: A \rightsquigarrow MA.$$

There's an invertible $m \times m$ matrix M such that MA is in rref.

Now we first recall that *row operations don't change the kernel:*

$$\ker(A) = \ker(MA).$$

However, row operations absolutely *do* change the image:

$$\mathrm{im}(A) \neq \mathrm{im}(MA).$$

Well, that's too bad! So what do we do to get out of trouble? We remember that the Rank-Nullity theorem only involves the *dimension* of the image, not the image itself.

And good news: *row operations don't change the* ***dimension*** *of the image.* So even though

$$\mathrm{im}(A) \neq \mathrm{im}(MA),$$

we still have

$$\dim(\mathrm{im}(A)) = \dim(\mathrm{im}(MA)).$$

(Why???)

Now, victory is ours! For any old $m \times n$ matrix A, we multiply on the left by an invertible $m \times m$ matrix M to get a matrix MA in rref, and we have

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = \dim(\ker(MA)) + \dim(\operatorname{im}(MA)) = n.$$

{*mic drop*}

Let's take a moment to imagine how our proof might have been different if we'd used column operations to get our matrix into rcef:

$$\textit{column operations}: A \rightsquigarrow AN,$$

where N is an invertible $n \times n$ matrix.

The point here is that *column operations don't change the image:*

$$\mathrm{im}(A) = \mathrm{im}(AN).$$

However, column operations absolutely *do* change the kernel:

$$\ker(A) \neq \ker(AN).$$

BUT, *column operations don't change the* ***dimension*** *of the kernel*:

$$\dim(\ker(A)) = \dim(\ker(AN)).$$

Using pure thought, tell me what the rank and nullity are of these matrices:

$$\begin{pmatrix} 5 & -15 \\ -2 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -138 \\ 5 & 1 & 75 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 6 & 3 \\ 5 & 1 & 50 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 9 & 9 \\ 1 & 1 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & 7 \\ -2 & 6 & 3 \\ -1 & 11 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix}$$