# 18.06.11: 'Nullspace'

Lecturer: Barwick

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Take some vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ and make them into the columns of an $n \times k$ matrix

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix}.$$

Multiplication by $A$ is a map $T_A \colon \mathbf{R}^k \longrightarrow V$ that carries $\hat{e}_i$ to $\vec{v}_i$.

(1) The vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ are *linearly independent* if and only if $T_A$ is injective.

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ *span* $V$ if and only if $T_A$ is surjective.

(3) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are a *basis* of $V$ if and only if $T_A$ is bijective.

Let's see why this works.

First, let's unpack what the injectivity of $T_A$ would mean. It's this condition for any $\vec{x}, \vec{y} \in \mathbf{R}^k$:

$$\text{if } A\vec{x} = A\vec{y}, \text{ then } \vec{x} = \vec{y}.$$

Defining $\vec{z} \coloneqq \vec{x} - \vec{y}$, we see that we want to show that

$$\text{if } A\vec{z} = \vec{0}, \text{ then } \vec{z} = \vec{0}.$$

(In other words, we're saying that the kernel of $A$ consists of just the zero vector!)

Now $A\vec{z}$ is a linear combination of the column vectors $\vec{v}_1, \ldots, \vec{v}_k$ with coefficients given by the components of $\vec{z}$. So the injectivity of $T_A$ is equivalent to the following:

$$\text{if } \sum_{i=1}^{k} z_i \vec{v}_i = \vec{0}, \text{ then } z_1 = \cdots = z_k = 0.$$

That's exactly what it means for $\vec{v}_1, \ldots, \vec{v}_k$ to be *linearly independent*.

Now let's unpack what the surjectivity of $T_A$ would mean. It's the condition that for any vector $\vec{w} \in V$, there exists a vector $\vec{x} \in \mathbf{R}^k$ such that $\vec{w} = A\vec{x}$. In other words, for any vector $\vec{w} \in V$, there exist numbers $x_1, \ldots, x_k$ such that

$$\vec{w} = \sum_{i=1}^{k} x_i \vec{v}_i.$$

That's exactly what it means for $\vec{v}_1, \ldots, \vec{v}_k$ to *span* the subspace $V$.

So we’ve proved our result: if

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix},$$

then

(1) $\vec{v}_1, \ldots, \vec{v}_k \in V$ are *linearly independent* if and only if $T_A$ is *injective*.

(2) $\vec{v}_1, \ldots, \vec{v}_k$ *span* $V$ if and only if $T_A$ is *surjective*.

(3) $\vec{v}_1, \ldots, \vec{v}_k$ are a *basis* of $V$ if and only if $T_A$ is *bijective*.

Now back to our motivating example: we've been given a system of linear equations

$$\vec{0} = A\vec{x},$$

where $A$ is an $m \times n$ matrix. To *solve* this equation is to find a basis for the *kernel* – AKA *nullspace* – of $A$.

In other words, the objective is to find a list of linearly independent solutions $\vec{v}_1, \ldots, \vec{v}_k$ such that any other solution can be written as a linear combination of these! The dimension of $\ker(A)$ – sometimes called the *nullity* of $A$ – is the number $k$.

There are two good ways of extracting a basis of the kernel. There's a way using *row operations*, and a way using *column operations*. You've been using these for a while already, but here's the way I think of these …

Suppose we can apply some row operations:

$$\left(\begin{array}{c|c} A & B \end{array}\right) \rightsquigarrow \left(\begin{array}{c|c} C & D \end{array}\right).$$

Here, $A$ and $C$ are $m \times n$ matrices, and $B$ and $D$ are $m \times p$ matrices. What this really means is that there's an invertible $m \times m$ matrix $M$ such that $MA = C$ and $MB = D$. (And it turns out that any $M$ can be built this way!)

That's why it works to solve equations: if in the end $C = I$, then $M = A^{-1}$, and $D = A^{-1}B$.

So you can use row operations to whittle your favorite matrix down, and then solve. This is nice because it's so familiar. Let's do a few examples together as a team.

First, how about

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}?$$

One more:

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

*Column operations* work exactly dual to row operations. (Just think of transposing, doing row operations, and transposing back!) So suppose we can apply some column operations:

$$\left(\frac{A}{B}\right) \rightsquigarrow \left(\frac{C}{D}\right).$$

Here, $A$ and $C$ are $m \times n$ matrices, and $B$ and $D$ are $p \times n$ matrices. What this really means is that there's an invertible $n \times n$ matrix $N$ such that $AN = C$ and $BN = D$. (And it turns out that any $N$ can be built this way!)

Why is that a good idea? Well, we're looking for vectors such that $A\vec{x} = \vec{0}$. So if we take

$$\left(\frac{A}{I}\right)$$

where $I$ is the $n \times n$ identity matrix, then we can start using column operations to get it to some

$$\left(\frac{C}{D}\right)$$

So $AD = C$. So if $C$ has a column of zeroes, then the corresponding column of $D$ will be a vector in the kernel. Furthermore, if you get $C$ into column echelon form, then the nonzero column vectors of $D$ lying under the zero columns of $C$ form a basis of $\ker(A)$.

Let's do this one again:

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Another:

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 2 & -8 \\ 0 & 1 & 5 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 7 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$