18.06.04: ‘Matrices’

Lecturer: Barwick

Wednesday 10 February 2016

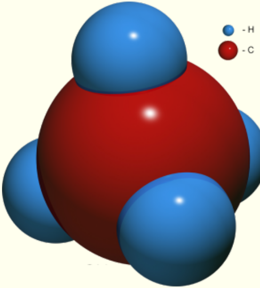


Figure 1: What are the angles between my bonds?

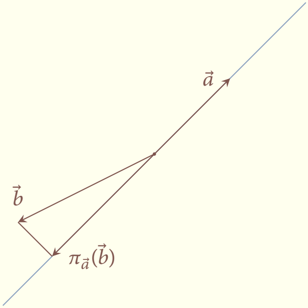
You were saying?

Exam 1 is a week from today

- It should be pretty simple.
- I will cover the course material through Friday’s lecture.
- You may not use any aids.

The *projection* of a vector $\vec{b}$ onto a vector $\vec{a}$ is the vector

$$\pi_{\vec{a}}(\vec{b}) \coloneqq (\hat{a} \cdot \vec{b})\hat{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}.$$



Matrices

Matrices are rectangular arrays of real numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Here, A has m rows and n columns. We say that A is an $m \times n$ matrix. We can think of A as a sequence of n vectors in $\mathbf{R}^m$:

$$A = \begin{pmatrix} \vec{A}^1 & \vec{A}^2 & \cdots & \vec{A}^n \end{pmatrix}$$

or as a sequence of m *row vectors* in $\mathbf{R}^n$:

$$A = \begin{pmatrix} \vec{A}_1 \\ \vec{A}_2 \\ \vdots \\ \vec{A}_m \end{pmatrix}.$$

The first thing that makes the notion of a matrix interesting is that you can multiply matrices by vectors. If

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbf{R}^n,$$

then we can write

$$A\vec{b} = b_1\vec{A}^1 + b_2\vec{A}^2 + \cdots + b_n\vec{A}^n = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{pmatrix}$$

Or, equivalently,

$$A\vec{b} := \begin{pmatrix} \vec{A}_1 \cdot \vec{b} \\ \vec{A}_2 \cdot \vec{b} \\ \vdots \\ \vec{A}_m \cdot \vec{b} \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{pmatrix}$$

One way to say what's going on here is to say that multiplication by an $m \times n$ matrix A is a *function*

$$T_A \colon \mathbf{R}^n \longrightarrow \mathbf{R}^m$$

that is defined so that

$$T_A(\vec{x}) = A\vec{x}.$$

This is an important way to think about matrices, and so we should pay close attention to this picture.

Let's think about this function applied to the unit vectors $\hat{e}_i$:

$$T_A(\hat{e}_i) = A\hat{e}_i = ?$$

$$T_A(\hat{e}_i) = A\hat{e}_i = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = \vec{A}^i.$$

$$A = \begin{pmatrix} A\hat{e}_1 & A\hat{e}_1 & \cdots & A\hat{e}_n \end{pmatrix}$$

Question. If $\theta \in [0, 2\pi)$, then we can form this 2×2 matrix:

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Describe the corresponding function $T_{R_\theta} : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ geometrically.

Armed with this, we can compile systems of m linear equations

$$\begin{aligned} v_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n; \\ v_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n; \\ &\vdots \\ v_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n; \end{aligned}$$

into the single equation $\vec{v} = A\vec{x}$, where $\vec{v} \in \mathbf{R}^n$ is a fixed vector, and $\vec{x} \in \mathbf{R}^m$ is a variable vector. Solving the system *is* solving the compiled equation.

But so what? It’s the same data, but a different format.

The point is that qualitative things about the system of linear equations can be extracted from qualitative things about the matrix.

Here’s a *diagonal* $n \times n$ matrix:

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \coloneqq \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix};$$

$$A\vec{b} = \begin{pmatrix} \lambda_1 b_1 \\ \lambda_2 b_2 \\ \vdots \\ \lambda_n b_n \end{pmatrix},$$

so equations $A\vec{x} = \vec{v}$ will be wicked easy to solve (uniquely??).

Only slightly less easy to solve will be what we get out of an $n \times n$ *upper triangular matrix*:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$A\vec{b} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{22}b_2 + a_{23}b_3 + \cdots + a_{2n}b_n \\ \vdots \\ a_{nn}b_n \end{pmatrix}.$$

Question. Suppose A an $m \times n$ matrix. When is a vector in $\mathbf{R}^m$ of the form $A\vec{b}$ for some vector $\vec{b} \in \mathbf{R}^n$?

Hint: We gave this formula

$$A\vec{b} = b_1\vec{A}^1 + b_2\vec{A}^2 + \cdots + b_n\vec{A}^n,$$

which exhibits $A\vec{b}$ as a linear combination of the column vectors of A …

Question. Suppose A an $m \times n$ matrix. When is the system of linear equations $A\vec{x} = \vec{v}$ *redundant*? That is, when is it the case that the equations provide the same constraints on the variable $\vec{x}$ that could be provided with fewer equations?

Hint: we have the dual formula

$$A\vec{b} := \begin{pmatrix} \vec{A}_1 \cdot \vec{b} \\ \vec{A}_2 \cdot \vec{b} \\ \vdots \\ \vec{A}_m \cdot \vec{b} \end{pmatrix}$$