18.06 Exam II: The Examining

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RECITATION: R666

GRADING	
1.	20 /20
2.	20 /20
3.	20 /20
4.	20 /20
5.	20 /20
TOTAL	/100

1. Yay or nay

For each of the following matrices, answer YES or NO: are they invertible? (You do *not* have to justify your answer.)

(a) $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ NO

(b) $\begin{pmatrix} 5 & 1 \\ 25 & 5 \end{pmatrix}$ NO

(c) $\begin{pmatrix} 1 & -1 \\ 6 & 5 \end{pmatrix}$ YES

(d) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ NO

(e) $\begin{pmatrix} 1 & 0 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{pmatrix}$ YES

(f) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ YES

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$ NO

(h) $\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 7 & 12 & 17 & 22 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 16 & 0 \\ 0 & 0 & 5 & 25 & 45 \end{pmatrix}$ YES

(i) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 5 \end{pmatrix}$ NO

(j) $\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$ YES

2. Solve

Find a basis for the space of solutions to the following system of linear equations in the seven variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7$:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 0 \\
x_2 + x_3 + x_4 + x_5 &= 0 \\
x_3 + x_4 + x_5 + x_6 &= 0 \\
x_4 + x_5 + x_6 + x_7 &= 0
\end{aligned}$$

Solution. We wish to compute the kernel of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It's almost in reduced row echelon form already. Clearing out the 1s over the pivots gives us

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

So if we set $x_5 = s$, $x_6 = t$, and $x_7 = u$, we can write everything in terms of s, t, and u:

$$\begin{aligned}
x_1 &= s \\
x_2 &= t \\
x_3 &= u \\
x_4 &= -s - t - u \\
x_5 &= s \\
x_6 &= t \\
x_7 &= u.
\end{aligned}$$

Thus our basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ □

3. Rank and file

Compute the rank of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

Solution. The rank is the dimension of the span of the column vectors $\vec{A}^1$, $\vec{A}^2$, $\vec{A}^3$, and $\vec{A}^4$. Simple arithmetic reveals that

$$\vec{A}^1 + \vec{A}^3 = 2\vec{A}^2$$

and

$$\vec{A}^2 + \vec{A}^4 = 2\vec{A}^3.$$

So, the columns can all be expressed as a linear combination of $\vec{A}^1$ and $\vec{A}^2$, and it is obvious that they are not multiples of each other. Hence the column space is 2-dimensional; that is, the rank is 2. □

4. Null at tea

Find a basis for the kernel of the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 \\ 1 & 2 & 5 & 14 & 42 & 132 \\ 2 & 5 & 14 & 42 & 132 & 429 \end{pmatrix}$$

Solution. Just for kicks, let's use column operations to get the kernel. We'll clear out each row of the top of the augmented matrix:

$$\left(\begin{array}{cccccc} 1 & 1 & 2 & 5 & 14 & 42 \\ 1 & 2 & 5 & 14 & 42 & 132 \\ 2 & 5 & 14 & 42 & 132 & 429 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 9 & 28 & 90 \\ 2 & 3 & 10 & 32 & 104 & 345 \\ \hline 1 & -1 & -2 & -5 & -14 & -42 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 5 & 20 & 75 \\ \hline 1 & -1 & 1 & 4 & 14 & 48 \\ 0 & 1 & -3 & -9 & -28 & -90 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & -1 & -6 & -27 \\ 0 & 1 & -3 & 6 & 32 & 135 \\ 0 & 0 & 1 & -5 & -20 & -75 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The first three columns on the top are a basis of the image, and the last three columns on the bottom are a basis for the kernel:

$$\left\{ \begin{pmatrix} -1 \\ 6 \\ -5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 32 \\ -20 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -27 \\ 135 \\ -75 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

□

5. Inversion invasion

Compute the inverse of this matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. Everyone's favorite way to invert a matrix is with row operations applied to $(A|I)$ (where A is our matrix):

$$\rightsquigarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 & 10 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 10 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 10 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & 4 & 10 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

So, magically,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad \square$$