18.06 MIDTERM EXAM 1

SOLUTIONS

(1) a) NO. b) YES. c) YES. c) NO. d) YES. e) NO.

(2) Multiple solutions; we need three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that

$$\frac{\mathbf{v}_i \cdot \mathbf{v}_j}{||\mathbf{v}_i||\,||\mathbf{v}_j||} = \frac{1}{2} \quad \text{for all } i \neq j.$$

Possible answers are:

- $(1,0,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$;
- $(1,0,0)$, $(1,\sqrt{3},0)$, $(3,\sqrt{3},2\sqrt{6})$;
- $\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{2}}{\sqrt{3}}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{2}}{\sqrt{3}}\right)$, $\left(0, \frac{\sqrt{3}}{3}, \frac{\sqrt{2}}{\sqrt{3}}\right)$;
- $(3,-\sqrt{3},2\sqrt{6})$, $(-3,-\sqrt{3},2\sqrt{6})$, $(0,2\sqrt{3},2\sqrt{6})$.

(3) Let us prove that the span of the columns of this matrix is $\mathbb{R}^n$. Denote the columns of the matrix by $\mathbf{v}_1, \ldots, \mathbf{v}_{100}$. Observe that $\frac{1}{99}\sum_{i=1}^{100} \mathbf{v}_i = (1,1,\ldots,1)$, so for all i we have $\mathbf{e}_i = \left(\frac{1}{99}\sum_{i=1}^{100} \mathbf{v}_i\right) - \mathbf{v}_i$. Thus, all $\mathbf{e}_i$ lie in the span of vectors $\mathbf{v}_i$, $i = 1, \ldots, 100$, so their span is indeed $\mathbb{R}^n$. Therefore, the vector $(1,2,\ldots,100)$ is also in the span of the columns and the system of linear equations has a solution.

To find out whether there is one solution or infinitely many solutions, we need to check whether the rows of the matrix span all of $\mathbb{R}^n$ (see lecture 5). The rows are the same set of vectors as the columns, so they do span all of $\mathbb{R}^n$, and there is a unique solution to this linear system.

Answer: There is a unique solution.

(4) Unit vectors on the two lines are $\pm\frac{1}{3}(1,-1,1,-1,1,-1,1,-1,1)$ and $\pm\frac{1}{3}(1,1,1,1,1,1,1,1,1)$ respectively. The absolute value of their dot product is $\frac{1}{9}$, so the angle is $\arccos\left(\frac{1}{9}\right)$.

Answer: $\arccos\left(\frac{1}{9}\right)$.

(5) The directions of the bonds are the directions from the origin to the vertices of the pentachoron, so it suffices to compute the angle between the vectors with the same coordinates as that of the vertices. Each of these vectors has norm $\frac{4}{\sqrt{5}}$, and dot product of any two is $-\frac{4}{5}$, therefore the angle between any two of them is $\arccos\left(-\frac{1}{4}\right)$.

Answer: $\arccos\left(-\frac{1}{4}\right)$.