18.06 Problem Set 8

Due Thursday, April 23, 2015, before 4:00 pm (sharp deadline) in Room E17-131

This homework has several questions to hand-in. Write down all details of your solutions, NOT JUST THE ANSWERS. Show your reasoning. Please staple the pages together and **clearly write your name**, your recitation section, and the name of your recitation instructor on the first page of the problem set.

Cooperation on problems is permitted, but all solutions must be written up independently and you must list your collaborators on the problem set. You should first try to solve each problem yourself, otherwise you will not learn much from hearing the solution.

This homework also has an online part and you can choose whether to do it (self-graded) on MITx with MATLAB or (human-graded) in Julia. Follow the instructions on course website.

Problem 1. Section 6.5, Problem 2, page 350.

Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compute the λ's. Find an $\boldsymbol{x}$ so that $\boldsymbol{x}^T A_1 \boldsymbol{x} < 0$, so A_1 fails the test.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$$

Solution. We can use any of the four tests. Let me try for demonstration to use a different test for each of the matrices. For A_1, we consider the energy test. For vector $(1, -1)$ the corresponding form $5x^2 + 12xy + y^2$ is zero, so A_1 is not positive definite. For A_2, consider the pivot test. The first pivot is negative, so the matrix is not positive-definite. For A_3 let us use the corner-determinants test. The determinant is zero, so the matrix is not positive-definite. The matrix A_4 can be repersented as $R^T R$, with independent columns in R: $R = \left[\begin{smallmatrix} 1 & 0 \\ 10 & 1 \end{smallmatrix}\right]$.

Problem 2. Section 6.5, Problem 20, page 352.

Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.
(b) The only positive definite projection matrix is $P = I$.
(c) A diagonal matrix with positive diagonal entries is positive definite.
(d) A symmetric matrix with a positive determinant might not be positive definite!

Solution. (a) The determinant is positive as all eigenvalues are positive. (b) All projection matrices except I are singular. (c) The diagonal entries of a diagonal matrix are its eigenvalues. (d) $A = -I$ has determinant equal to 1 when n is even, but it is not a positive-definite matrix.

Problem 3. Section 6.5, Problem 28, page 353.

Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A
(b) the eigenvalues of A
(c) the eigenvectors of A
(d) a reason why A is symmetric positive definite.

Solution. (a) The determinant of A is the product of the determinants: $1 \cdot 10 \cdot 1 = 10$. (b) The eigenvalues of A are the same as the eigenvalues of its diagonalization, that is 2 and 5. (c) The eigenvectors are the columns of S: $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$. (d) The eigenvalues are positive.

Problem 4. Section 6.6, Problem 20, page 362.

Why are these statements all true?

(a) If A is similar to B then A^2 is similar to B^2.
(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).
(c) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ is similar to $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 4 \end{smallmatrix}\right]$
(d) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ is not similar to $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$
(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same**. In this case $M =$ ___________.

Solution. (a) If $A = M^{-1}BM$, then $A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$. So A^2 is similar to B^2. (b) I_2 and $-I_2$ are not similar, but their squares are I_2. (c) Both matrices have the same distinct eigenvalues 3 and 4, so they are similar. (d) The second matrix has only one eigenvector, so it is not diagonalizable. (e) The change of bases is made by swapping rows 1 and 2. In 2-d case $M = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.