

## 18.06 PROBLEM SET 7 - SOLUTIONS

### Problem 1.

We have that  $\det(A - \lambda I)$  is  $-\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda^2 - \lambda + 2)$ , so that the eigenvalues are  $0, -1, 2$ . Because the eigenvalues are all distinct, the matrix is diagonalizable. Looking at the nullspace of  $A - \lambda I$  for  $\lambda = 0, -1, 2$  we find that the eigenvectors are multiples of  $(-1, 1, 1)$ ,  $(1, -1, -1)$  and  $(2, 1, 1)$ . If we call  $S$  the matrix that has these three vectors as columns, we have

$$A = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} S^{-1}$$

Notice that this decomposition depends on the choice of the order on the eigenvalues and which actual eigenvector we are picking, but all choices will lead to the same result of course. In our specific case we have

$$S = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad S^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 3 & -3 \\ 2 & -2 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

so that for  $k \geq 1$  we have

$$A^k = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} S^{-1}$$

and the result is

$$\frac{1}{6} \begin{bmatrix} 2^{k+2} + (-1)^k 2 & 2^{k+1} + (-1)^{k+1} 2 & 2^{k+1} + (-1)^{k+1} 2 \\ 2^{k+1} + (-1)^{k+1} 2 & 2^k + (-1)^k 2 & 2^k + (-1)^k 2 \\ -2^{k+1} + (-1)^{k+1} 2 & 2^k + (-1)^k 2 & 2^k + (-1)^k 2 \end{bmatrix}$$

This formula does not work when  $k$  is zero!

### Problem 2.

(a) We have that as  $\mathbf{u}$  solves the equations

$$2u_1 u'_1 + 2u_2 u'_2 + 2u_3 u'_3 = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2)$$

and all terms cancel.

(b) We have

$$Q = e^{At} = \sum_{n \geq 0} \frac{(tA)^n}{n!}$$

so that transposing everything and using the fact that  $A$  is skew-symmetric

$$Q^T = \sum_{n \geq 0} \frac{((tA)^T)^n}{n!} = \sum_{n \geq 0} \frac{(-tA)^n}{n!} = e^{-At}.$$

So we have that  $Q^T Q = e^{-At} e^{At} = I$ , hence  $Q$  is orthogonal.

**Problem 3.**

Let  $\lambda$  be a complex eigenvalue of  $B$ . From the fact  $B^4 = I$  we obtain that  $\lambda^4 = 1$ , so  $\lambda = \pm 1, \pm i$ . On the other hand,  $\det(B - \lambda I)$  is a second degree polynomial which has real coefficients because the entries of  $B$  are real. In particular it has two zeroes, and if  $\lambda$  is a zero then also  $\bar{\lambda}$  is a zero. This implies that only four pairs of eigenvalues can actually occur, namely

$$(1, 1) \quad (1, -1) \quad (-1, -1) \quad (i, -i).$$

Finally, these are realized for examples by the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which all satisfy  $B^4 = I$ .