18.06 Problem Set 5

Due Thursday, March 19, 2015, before 4:00 pm (sharp deadline) in Room E17-131

This homework has 4 questions to hand-in. Write down all details of your solutions, NOT JUST THE ANSWERS. Show your reasoning. Please staple the pages together and **clearly write your name**, your recitation section, and the name of your recitation instructor on the first page of the problem set.

Cooperation on problems is permitted, but all solutions must be written up independently and you must list your collaborators on the problem set. You should first try to solve each problem yourself, otherwise you will not learn much from hearing the solution.

This homework also has an online part and you can choose whether to do it (self-graded) on MITx with MATLAB or (human-graded) in Julia. Follow the instructions on course website.

Problem 1. We are given three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in the plane $\mathbb{R}^2$, and we want to find a line $y = C + Dx$ which minimizes the error term $E = (C + Dx_1 - y_1)^2 + (C + Dx_2 - y_2)^2 + (C + Dx_3 - y_3)^2$. Suppose that x_1, x_2, x_3 are distinct.

(a) Check that the equations $\frac{\partial E}{\partial C} = 0$ and $\frac{\partial E}{\partial D} = 0$ are linear with regard to C and D. Write this system of equations as $M \begin{bmatrix} C \\ D \end{bmatrix} = \mathbf{v}$, where M is a 2×2 matrix.

(b) Let $A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Calculate $A^T A$ and $A^T \mathbf{b}$, and conclude that E is minimized when $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T \mathbf{b}$.

(c) Show that $A^T A$ is invertible and our desired line $y = C + Dx$ always exists uniquely.

Problem 2. Suppose that vectors $q_1, q_2, \cdots, q_n$ in $\mathbb{R}^m$ are orthonormal.

(a) Let $c_1, c_2, \cdots, c_n$ be real numbers. What is $||c_1 q_1 + c_2 q_2 + \cdots + c_n q_n||^2$?

(b) Show that $q_1, q_2, \cdots, q_n$ are linearly independent.

Problem 3. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$.

(a) Find Q and R such that $A = QR$ by Gram-Schmidt process on columns of A.

(b) Let $B = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, a matrix obtained by exchanging the first two columns of A. Find Q' and R' such that $B = Q'R'$ by Gram-Schmidt process on columns of B.

Problem 4. Suppose that a function $f(x)$ is defined from 0 to 2π as follows: $f(x) = 1$ for $0 \leq x \leq \pi$ and $f(x) = 0$ for $\pi < x < 2\pi$. Compute the Fourier coefficients a_k and b_k.