18.06 Problem Set 4 - SOLUTIONS

Problem 1. Section 3.6, Problem 28, page 194.

Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

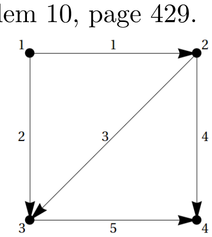
Solution.

(a) B: Matrix B has rank 2. Row 1 and 2 are a basis for the row space of B. The left nullspace $B^T y = 0$ has 6 special solutions. They all can be described as vectors containing a -1 and 1 separated by a zero: $(-1, 0, 1, 0, 0, 0, 0, 0)$, $(0, -1, 0, 1, 0, 0, 0, 0)$, $(0, 0, -1, -0, 1, 0, 0, 0)$, $(0, 0, 0, -1, 0, 1, 0, 0)$, $(0, 0, 0, 0, -1, 0, 1, 0)$, and $(0, 0, 0, 0, 0, -1, 0, 1)$.

(b) C: Let us assume that $p \neq 0$. Matrix C has rank 2. Row 1 and 2 are a basis for the row space of C. The left nullspace has $(-1, 0, 0, 0, 0, 0, 0, 1)$ and $(0, -1, 0, 0, 0, 0, 1, 0)$ and columns 3, 4, 5, 6 of I. We can use the standard procedure to find the nullspace of C, but it is more fun to try to exploit the structure of the matrix C. Elements of the nullspace correspond to the dependencies of the columns. Repeated columns allow us to generate three vectors in the nullspace $(-1, 0, 0, 0, 0, 0, 0, 1)$, $(0, -1, 0, 0, 0, 0, 1, 0)$, and $(0, 0, -1, 0, 0, 1, 0, 0)$. We need three more vectors. We can try to find vectors with 3 non-zero elements. For example, we can try to express the columns related to rook(r), knight(n), and bishop(b) through queen(q) and king(k). Every element in the nullspace has to have the sum of its coordinates equal to 0. So we can find the following vectors $(k - q, 0, 0, r - k, q - r, 0, 0, 0)$, $(0, k - q, 0, n - k, q - n, 0, 0, 0)$, and $(0, 0, k - q, b - k, q - b, 0, 0, 0)$.

If $p = 0$, matrix C has rank 1. Row 1 is the bases for the row space. The left nullspace has $(-1, 0, 0, 0, 0, 0, 0, 1)$ and columns 2, 3, 4, 5, 6, 7 of I. There are many ways to generate the nullspace. For example, we can calculate vectors with non-zero elements in the first column and another column: $(-n, r, 0, 0, 0, 0, 0)$, $(-b, 0, r, 0, 0, 0, 0, 0)$, and so on.

Problem 2. Section 8.2, Problem 10, page 429.



Write down the 5 by 4 incidence matrix A for the square graph with two loops. Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

Solution. The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The nonzero rows of U keep edges 1, 2, 4. Other spanning trees are formed by edges: (1,2,5), (1,3,4), (1,3,5), (1,4,5), (2,3,4), (2,3,5) and (2,4,5).

Problem 3. Section 8.2, Problem 9, page 429.

For the square graph above, find two requirements on the b's for the five differences $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$, $x_4 - x_2$, $x_4 - x_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's ____________ around the two ____________ in the graph.

Solution. Elimination on $Ax = b$ always leads to $y^T b = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$. This is Kirchhoff's *Voltage* Law around the two *loops*.

Problem 4. Section 8.2, Problem 17, page 430.

Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?
(b) What condition on f makes it possible to solve $A^T y = f$?
(c) The diagonal entries of $A^T A$ give the number of edges into each node. What is the sum of those diagonal entries?

Solution.

(a) The nullspace of an incidence matrix of a connected graph is always one-dimensional. Hence, the rank of this matrix is 8, and there are 8 independent columns.
(b) f must belong to the column space of A^T, which is the same as row space of A. A vector is in the row space if and only if it is perpendicular to $(1, 1, 1, 1, 1, 1, 1, 1, 1)$. Or in other words, if its coefficients sum to zero.
(c) By "the number of edges into each node" the problem means the number of edges coming in and going out. As each edge contributes 2 ends to the sum, the total is $12 \cdot 2 = 24$.

Problem 5. Section 4.1, Problem 28, page 205.

Why is each of these statements false?

(a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x+y+z = 0$ and $x+y-2z = 0$ are orthogonal subspaces.
(b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.
(c) Two subspaces that meet only at the zero vector are orthogonal.

Solution.

(a) Two planes in 3D can not be orthogonal to each other as the sum of their dimensions is greater than 3.

(b) The first subspace is a 2-dimensional subspace in a 5-dimensional space. Its orthogonal complement must be 3-dimensional, so it can not be spanned by two vectors.

(c) For example, consider two lines $x = 0$ and $x + y = 0$ on the plane. These are two lines that intersect at the zero vector, but they are not perpendicular to each other, so they are not orthogonal subspaces.

Problem 6. Section 4.1, Problem 30, page 205.

Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. So $\boldsymbol{N}(A)$ contains $\boldsymbol{C}(B)$. Prove from the dimensions of $\boldsymbol{N}(A)$ and $\boldsymbol{C}(B)$ that $\text{rank}(A) + \text{rank}(B) \leq 4$.

Solution. When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore, the dimension of $\boldsymbol{C}(B) \leq$ dimension of $\boldsymbol{N}(A)$. This means $\text{rank}(B) \leq 4 - \text{rank}(A)$.