18.06 Problem Set 3 - SOLUTIONS

Problem 1.

(a) We are looking for a $n \times 5$ reduced matrix R which has 3 dimensional nullspace. This implies that R has rank 2. We then start by looking for a 2×5 matrix. We first need to identify the positions of the pivots or, equivalently, of the free variables. The latter are the variables that we are allowed to set to any value we want, so if we consider the matrix

$$B = \begin{pmatrix} 1 & 0 & 5 & 0 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

whose rows are the $\mathbf{s}_i$, they correspond to the rightmost triple of columns which are independent. The fifth and forth columns are independent, while the third is a combination of them. Finally the second row is independent of them, so the free variables are x_2, x_4 and x_5 and the matrix has the form

$$R = \begin{pmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \end{pmatrix}$$

where the $*$ denote numbers to be determined. To find these numbers, we impose each row to be orthogonal to each of the $\mathbf{s}_i$, and the final result is

$$R = \begin{pmatrix} 1 & 1/2 & 0 & -2 & -1 \\ 0 & 0 & 1 & -2 & -5 \end{pmatrix}$$

To find the $n \times 5$ matrix, we just add rows of zeros at the bottom.

(b) From now on we fix $n \geq 2$. Another matrix with the desired property can be obtained by swapping the first two rows, or in general by doing any row manipulation (subtracting a multiple of a row from another one, multiplying rows by non zero numbers).

(c) The answer is invertible $n \times n$ matrix. Suppose A and A' both have the desired property. Then their reduced form is the R (with possible more rows of zeros) described above, and each of them can be reduced to this form by row operations. In particular we can obtain A' from A by row operations, hence by multiplying A an invertible matrix. Similarly if M is invertible, A and MA have the same nullspace. Hence all the matrices with the desired property are obtained one from the other by multiplying by an invertible matrix.

Problem 2.

The number of special solutions is the same as the number of the free variables. So the sum the number of pivots and the number of special solutions is the total number of columns, n. The pivot columns are a basis for the column space, while the special solutions are a basis for the nullspace. So this identity implies that the sum of the dimensions of the column space and the nullspace is n.

Problem 3.

(a) The fact that the there exists $\mathbf{b} \in R^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ doesn't have a solution implies that the column space of A is not the whole R^m, so its dimension r is strictly smaller than m. We also know that $r \leq n$, because there are at most n pivots. There are no inequalities that always hold between m and n. This can be seen by adding rows and columns of zeros to our matrix.
(b) Saying that the only solution of $A^T\mathbf{y} = 0$ is $\mathbf{y} = 0$ is the same as saying that the rows of A are independent, so A has full row rank. As there are m of them, this implies that $r = m$, which is in contradiction with the discussion above.

Problem 4.
By doing row operations (second minus twice the first, third minus the second, first minus the second) we end up solving the problem in reduced form

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

and the free variables are y and t. By setting them both to zero, we find the particular solution $(1/2, 0, 1/2, 0)$. Every other solution is obtained by adding an element in the nullspace. To find the special solutions, we set for example $(y, t) = (1, 0)$ which gives $(-3, 1, 0, 0)$ and $(y, t) = (1, 0)$ which gives $(0, 0, -2, 1)$. Hence the general solution can be written in the form

$$\begin{bmatrix} 1/2 - 3\lambda \\ \lambda \\ 1/2 - 2\mu \\ \mu \end{bmatrix}$$

for some numbers λ, μ.