18.06 Problem Set 2 - SOLUTIONS

Problem 1.

(I) The top left entry is 0, so we need to permute the first two rows, so we take

$$P = \begin{pmatrix} 0,1,0 \\ 1,0,0 \\ 0,0,1 \end{pmatrix}.$$

Then

$$PA = \begin{pmatrix} 1,0,1 \\ 0,1,1 \\ 2,3,4 \end{pmatrix} = \begin{pmatrix} 1,0,0 \\ 0,1,0 \\ 2,3,1 \end{pmatrix} \begin{pmatrix} 1,0,1 \\ 0,1,1 \\ 0,0,-1 \end{pmatrix} = LU.$$

(II) After the first step, we obtain the matrix

$$\begin{pmatrix} 1,2,0 \\ 0,0,1 \\ 0,-1,1 \end{pmatrix}.$$

Thus, we need to permute the last two rows, so we take

$$P = \begin{pmatrix} 1,0,0 \\ 0,0,1 \\ 0,1,0 \end{pmatrix}.$$

And then

$$PA = \begin{pmatrix} 1,2,0 \\ 1,1,1 \\ 2,4,1 \end{pmatrix} = \begin{pmatrix} 1,0,0 \\ 1,1,0 \\ 2,0,1 \end{pmatrix} \begin{pmatrix} 1,2,0 \\ 0,-1,1 \\ 0,0,1 \end{pmatrix} = LU.$$

Problem 2.

(a) If S and T are the same line, then since a line throught the origin is a subspace, $S \cup T = S$ is as well. If they are different lines, then they do not form a subspace, since if we take a nonzero vector from one, and from the other, their sum will not be on either line.

(b) Since the two lines both go through the origin, there is a plane through the origin that contains both lines, this is the smallest possible subspace containing both lines.

(c) Take the set of vectors that arise as the sum of a vector from S and a vector from T. Clearly any subspace containing both S and T must contain all such vectors, and one can check that this describes a subspace (assuming S and T both are).

Problem 3.

(a) v being in the column space of AB means that there is an x such that $ABx = v$. But then if we take $y = Bx$, then $Ay = ABx = v$, so v is in the column space of A.

(b) If we take $A = \begin{pmatrix} 1,0,0 \\ 0,0,0 \\ 0,0,0 \end{pmatrix}$ and $B = \begin{pmatrix} 0,0,0 \\ 0,0,0 \\ 0,0,1 \end{pmatrix}$, then $AB = 0$, so its column space is Z (only the zero the vector), but the column space of A is the line through $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(c) We can take A to be the 3×3 matrix that permutes the first two coordinates, so

$$A = \begin{pmatrix} 0,1,0 \\ 1,0,0 \\ 0,0,1 \end{pmatrix}.$$

Then if $C(B)$ is a line generated by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then $C(AB)$ is generated by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

So we can take $A = \begin{pmatrix} 0,1,0 \\ 1,0,0 \\ 0,0,1 \end{pmatrix}$, $B = \begin{pmatrix} 1,1,1 \\ 0,0,0 \\ 0,0,0 \end{pmatrix}$, and $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Problem 4.

We have n choices for where to put 1 in the first column, $n-1$ for the second, and so on. This gives $n!$.

The number of various types of matrices is

(a) 1 (just the identity).
(b) 0, if you move one entry, you must move another!
(c) 6, we can choose the pair of entries in 6 ways, and then all we can do is switch them.
(d) 8, we choose an element that we want to remain fixed, this can be done in four ways, then we have two choices for how to move the other three.
(e) 9, we can either switch two pairs of elements, which we can do in three ways, or we can cyclically switch around the elements, which we can do in six ways.