18.06 Final Exam **Professor Strang** **May 18, 2015**

SOLUTIONS

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Grading

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Thank you for taking 18.06! I hope you have a wonderful summer!

EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form $R = \text{rref}(A)$ for this matrix A :

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution. We have

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last matrix is the RREF.

(b) Find a basis for the column space $C(A)$.

Solution. We can see that the pivot columns are columns 1 and 3, so these columns from the *original* matrix form a basis,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Find all solutions (and first tell me the conditions on b_1, b_2, b_3 for solutions to exist!).

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution. We can see that we need $b_2 = b_3$. First, let us find a particular solution. Since x_2, x_4 are free variables, we can set them to 0, and then we can solve to get

$$\begin{pmatrix} b_1 - b_2 \\ 0 \\ b_2 \\ 0 \end{pmatrix}.$$

Now, we need a basis for the nullspace, the special solutions. Setting each free variable to 1 and the other to 0, we obtain the special solutions

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\1\end{pmatrix}$$

So, the general solutions are given by vectors

$$\begin{pmatrix}b_1 - b_2\\0\\b_2\\0\end{pmatrix} + c_1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\-2\\1\end{pmatrix}$$

2. (a) What is the 3 by 3 projection matrix P_a onto the line through $a = (2, 1, 2)$?

Solution.

$$P_a = \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}}{\begin{pmatrix} 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} = \frac{1}{9} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

(b) Suppose P_v is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_a P_v$ (product of 2 projections)

Solution. $P_a P_v v = P_a v = \frac{5}{9} a$ and so $a \in C(P_a P_v) \subset C(P_a)$. Since $C(P_a)$ is spanned by a, a basis for $C(P_a P_v)$ is given by $\{a\}$.

3. Suppose I give you an orthonormal basis $q_1, \ldots, q_4$ for $\mathbf{R}^4$ and an orthonormal basis $z_1, \ldots, z_6$ for $\mathbf{R}^6$. From these you create the 6 by 4 matrix $A = z_1 q_1^T + z_2 q_2^T$.

(a) Find a basis for the nullspace of A.
Solution. The matrix has SVD ZJQ^T where J is the 6 by 4 matrix with diagonal entries $(1, 1, 0, 0)$. This means that its nullspace consists of the q's in columns of Q corresponding to zero singular values, which is q_3, q_4.

(b) Find a particular solution to $Ax = z_1$ and find the complete solution.
Solution. One particular solution to Ax_1 is q_1, since $(z_1 q_1^T) q_1 = z_1(q_1^T z_1) = z_1(q_1 \cdot q_1) = z_1$, by and $(z_2 q_2^T) q_1 = z_2(q_2^T q_1) = z_1(q_2 \cdot q_1) = 0$ by orthonormality of q_i. The complete solution is obtained by adding an element of the nullspace, i.e. a linear combination of basis vectors of the nullspace: $q_1 + cq_2 + dq_4$ for scalars c, d.

(c) Find $A^T A$ and find an eigenvector of $A^T A$ with $\lambda = 1$.
Solution. $A^T A = (q_1 z_1^T + q_2 z_2^T)(z_1 q_1^T + z_2 q_2^T)$. Expanding and reparenthezising gives $A^T A = q_1(z_1^T z_1) q_1^T + q_1(z_1^T z_2) q_2^T + q_2(z_2 z_1^T) q_2^T + q_2(z_2 z_2^T) q_2^T$. In every term, the parenthesized scalar in the middle is a dot product: $z_1 \cdot z_2 = 0$ for the middle two terms and 1 for the first and fourth terms, leaving $A^T A = q_1 q_1^T + q_2 q_2^T$. We see that $A^T A q_1 = q_1(q_1 \cdot q_1) + q_2(q_2 \cdot q_1) = q_1$ and, for the same reason, $A^T A q_2 = q_2$. So q_1 and q_2 (or any nonzero linear combination) are all eigenvectors with eigenvalue 1.

4. Symmetric positive definite matrices H and orthogonal matrices Q are the most important. Here is a great theorem: *Every square invertible matrix A can be factored into $A = HQ$.*

(a) Start from $A = U\Sigma V^T$ (the SVD) and *choose* $Q = UV^T$. Find the other factor H so that $U\Sigma V^T = HQ$. Why is your H symmetric and why is it positive definite?
Solution. By definition we need $U\Sigma V^T = A = HQ = HUV^T$ so we get by inverting U and V^T (which are orthogonal hence invertible) that $H = U\Sigma U^{-1}$. The last item can also be written as $U\Sigma U^T$ because U is orthogonal. This matrix is symmetric because $H^T = (U\Sigma U^T)^T = U\Sigma^T U^T = H$ as Σ is diagonal so it is equal to its own transpose. To see that it is positive definite we can use the eigenvalue test: the eigenvalues of H are given by the diagonal elements of Σ, i.e. the singular values of A. They are all *nonnegative* because they are the eigenvalues of A^TA, and they cannot be zero because A is invertible by assumption. Hence the eigenvalues of H are all positive.

(b) Factor this 2 by 2 matrix into $A = U\Sigma V^T$ and then into $A = HQ$:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = U\Sigma V^T = HQ$$

Solution. We have $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix}$ so the singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = \sqrt{2}$ and the corresponding eigenvectors are $v_1 = (0, 1)$ and $v_2 = (1, 0)$ so that $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We then have

$$u_1 = Av_1/\sigma_1 = (1/\sqrt{2}, 1/\sqrt{2}) \qquad u_2 = Av_2/\sigma_2 = (1/\sqrt{2}, -1/\sqrt{2}),$$

so the SVD is

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Finally

$$H = U\Sigma U^T = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2\sqrt{2} \end{bmatrix} \qquad Q = UV^T \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

5. (a) Are the vectors $(0,1,1),(1,0,1),(1,1,0)$ independent or dependent?

Solution. These vectors are independent. One way to see this is that

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2 \neq 0$$

(b) Suppose T is a linear transformation with input space = output space = $\mathbf{R}^3$. We have a basis u, v, w for $\mathbf{R}^3$ and we know that $T(u) = v + w, T(v) = u + w, T(w) = u + v$. Describe the transformation T^2 by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.

Solution. We have

$$T^2(u) = T(v + w) = T(v) + T(w) = 2u + v + w$$

$$T^2(v) = T(u + w) = T(u) + T(w) = u + 2v + w$$

$$T^2(w) = T(u + v) = T(u) + T(v) = u + v + 2w$$

Note that this means that in the basis (u, v, w), the matrix of T^2 is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

6. Suppose A is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors x_1, x_2, x_3.

(a) What is the rank of A? Describe all vectors in its column space $C(A)$.

Solution. Vectors x_1, x_2, and x_3 are independent. Any vector y in $\mathbf{R}^3$ can be represented as a linear combination of the eigenvectors: $y = ax_1 + bx_2 + cx_3$. Applying A we get $Ay = bx_2 - cx_3$. Thus x_2 and x_3 form a basis in the column space and the rank of A is 2.

(b) How would you solve $du/dt = Au$ with $u(0) = (1, 1, 1)$?

Solution. By the formula $u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$, where λ_i are eigenvalues and x_i the corresponding eigenvectors. We are given λ_i and x_i, so we can plug them in to get: $u(t) = c_1 e^{0t} x_1 + c_2 e^t x_2 + c_3 e^{-t} x_3 = c_1 x_1 + c_2 e^t x_2 + c_3 e^{-t} x_3$. To find the coefficients c_1, c_2, and c_3, we need to use the initial conditions, that is to solve the equation: $u(0) = (1, 1, 1) = c_1 x_1 + c_2 x_2 + c_3 x_3$.

(c) What are the eigenvalues and determinant of e^A?

Solution. The eigenvalues of e^A are the same as the eigenvalues of e^Λ, where Λ is the diagonalization of A. Therefore, the eigenvalues of e^A equal e to the power of the eigenvalues of A: $e^0 = 1$, $e^1 = e$ and $e^{-1} = 1/e$. The determinant is the product of the eigenvalues and is equal to $1 \cdot e \cdot 1/e = 1$.

7. (a) Find a 2 by 2 matrix such that

$$A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix} \text{ and also } A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

or say why such a matrix can't exist.

Solution. $A=\begin{pmatrix}1&1\\4/3&4/3\end{pmatrix}$ is the 2 by 2 matrix such that $A\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$ and $A\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$. One way to arrive at A is to let $B=\begin{pmatrix}3&3\\4&4\end{pmatrix}$ be the matrix which sends the standard basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ both to $\begin{pmatrix}3\\4\end{pmatrix}$ and let $C=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ be the change of basis matrix which sends the standard basis vectors to $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$. Then $A=BC^{-1}$.

(b) The columns of this matrix H are orthogonal but not orthonormal:

$$H=\begin{bmatrix}1&1&1&1\\-1&1&1&1\\0&-2&1&1\\0&0&-3&1\end{bmatrix}$$

Find H^{-1} by the following procedure. First multiply H by a diagonal matrix D that makes the columns orthonormal. Then invert. Then account for the diagonal matrix D to find the 16 entries of H^{-1}.

Solution. To normalize the columns of H, we let D be the diagonal matrix with diagonal entries $1/\sqrt{2}$, $1/\sqrt{6}$, $1/\sqrt{12}$, and $1/2$, and we multiply H by D on the right: $H'=HD$. Because H' is an orthogonal matrix, $H'^{-1}=H'^T$. Then $H^{-1}=D(HD)^{-1}=DH'^T$.

Computing, we obtain $H^{-1}=\begin{pmatrix}1/2&-1/2&0&0\\1/6&1/6&-1/3&0\\1/12&1/12&1/12&-1/4\\1/4&1/4&1/4&1/4\end{pmatrix}$.

8. (a) Factor this symmetric matrix into $A = U^T U$ where U is upper triangular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution. By applying row operations we find the factorization $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $L = U^T$.

(b) Show by two different tests that A is symmetric positive definite.
Solution. Unfortunately it is hard to compute the eigenvalues explicitly, but nevertheless one can apply one of these tests:

i. $A = U^T U$ for U invertible;
ii. the energy test, $x^T A x = x U^T U x = \|Ux\|^2 \geq 0$ of $x \neq 0$ because U is invertible;
iii. the pivots of A are the pivots of U which are all positive;
iv. the upper left determinants of A are all 1 hence positive;
v. the eigenvalues satisfy the equation $-(\lambda^3 - 6\lambda^2 + 5\lambda - 1)$ which cannon be zero for negative λ by checking the signs in the sum.

(c) Find and explain an upper bound on the eigenvalues of A. Find and explain a (positive) lower bound on those eigenvalues if you know that

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are positive and they sum to the trace, which is 6, so they can be at most 6. The inverses of the eigenvalues $1/\lambda_1, 1/\lambda_2, 1/\lambda_3$ are the eigenvalues of A^{-1}, which has trace 5, so this tells us that each of the λ_i is at least $1/5$.

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