18.06 Final Exam   Professor Strang   May 18, 2015

Your PRINTED Name is: ______________________________

Please CIRCLE your section:

R01	T10	26-302	Dmitry Vaintrob
R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
R04	T11	26-322	Francesco Lin
R05	T11	26-328	Laszlo Lovasz
R06	T12	36-144	Michael Andrews
R07	T12	26-302	Netanel Blaier
R08	T12	26-328	Laszlo Lovasz
R09	T1pm	26-302	Sungyoon Kim
R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading

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Thank you for taking 18.06! I hope you have a wonderful summer!

EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form $R = \text{rref}(A)$ for this matrix A :

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(b) Find a basis for the column space $C(A)$.

(c) Find all solutions (and first tell me the conditions on b_1, b_2, b_3 for solutions to exist!).

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

2. (a) What is the 3 by 3 projection matrix P_a onto the line through $a = (2, 1, 2)$?

(b) Suppose P_v is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_a P_v$ (product of 2 projections)

3. Suppose I give you an orthonormal basis $q_1, \ldots, q_4$ for $\mathbf{R}^4$ and an orthonormal basis $z_1, \ldots, z_6$ for $\mathbf{R}^6$. From these you create the 6 by 4 matrix $A = z_1 q_1^T + z_2 q_2^T$.

(a) Find a basis for the nullspace of A.

(b) Find a particular solution to $Ax = z_1$ and find the complete solution.

(c) Find $A^T A$ and find an eigenvector of $A^T A$ with $\lambda = 1$.

4. Symmetric positive definite matrices H and orthogonal matrices Q are the most important. Here is a great theorem: *Every square invertible matrix A can be factored into $A = HQ$.*

(a) Start from $A = U\Sigma V^T$ (the SVD) and *choose* $Q = UV^T$. Find the other factor H so that $U\Sigma V^T = HQ$. Why is your H symmetric and why is it positive definite?

(b) Factor this 2 by 2 matrix into $A = U\Sigma V^T$ and then into $A = HQ$:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = U\Sigma V^T = HQ$$

5. (a) Are the vectors $(0,1,1),(1,0,1),(1,1,0)$ independent or dependent?

(b) Suppose T is a linear transformation with input space $=$ output space $= \mathbf{R}^3$. We have a basis u, v, w for $\mathbf{R}^3$ and we know that $T(u) = v + w, T(v) = u + w, T(w) = u + v$. Describe the transformation T^2 by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.

6. Suppose A is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors x_1, x_2, x_3.

(a) What is the rank of A? Describe all vectors in its column space $C(A)$.

(b) How would you solve $du/dt = Au$ with $u(0) = (1, 1, 1)$?

(c) What are the eigenvalues and determinant of e^A?

7. (a) Find a 2 by 2 matrix such that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and also } A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

or say why such a matrix can't exist.

(b) The columns of this matrix H are orthogonal but not orthonormal:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

Find H^{-1} by the following procedure. First multiply H by a diagonal matrix D that makes the columns orthonormal. Then invert. Then account for the diagonal matrix D to find the 16 entries of H^{-1}.

8. (a) Factor this symmetric matrix into $A = U^T U$ where U is upper triangular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Show by two different tests that A is symmetric positive definite.

(c) Find and explain an upper bound on the eigenvalues of A. Find and explain a (positive) lower bound on those eigenvalues if you know that

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Scrap Paper