18.06 Exam II Solutions Professor Strang April 10, 2015

1. (33 points) Suppose we measure $b = 0, 0, 0, 1, 0, 0, 0$ at times $t = -3, -2, -1, 0, 1, 2, 3$.

(a) To fit these 7 measurements by a straight line $C + Dt$, what 7 equations $Ax = b$ would we want to solve?

Solution. We want to solve the following 7 equations: $C - 3D = 0$, $C - 2D = 0$, $C - D = 0$, $C = 1$, $C + D = 0$, $C + 2D = 0$, and $C + 3D = 0$.

(b) Find the least squares solution $\widehat{x} = (\widehat{C}, \widehat{D})$.

Solution. First we need to find the projection of b onto the plane generated by two vectors: $(1, 1, 1, 1, 1, 1, 1)$ and $(-3, -2, -1, 0, 1, 2, 3)$. As b is perpendicular to the second vector, we only need to find the projection of b on the line generated by the first vector, which is $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$. Now we need to solve the seven equations: $C - 3D = 1/7$, $C - 2D = 1/7$, $C - D = 1/7$, $C = 1/7$, $C + D = 1/7$, $C + 2D = 1/7$, and $C + 3D = 1/7$, and $C = 1/7$ and $D = 0$.

Alternatively, we can denote by A the matrix that has these two vectors as its two columns, then $A^T A = \left[\begin{smallmatrix} 7 & 0 \\ 0 & 28 \end{smallmatrix}\right]$ and $A^T b = (1, 0)$. The two equations corresponding to $A^T A\widehat{x} = A^T b$ are $7C = 1$ and $28D = 0$, resulting in the same solution $C = 1/7$ and $D = 0$.

(c) The projection of that vector b in $\mathbf{R}^7$ onto the column space of A is what vector p?

Solution. If we used the first method above, we already calculated the projection as $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$. If we used the second method, the projection is $A\widehat{x} = (1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)^T$.

2. (34 points) Suppose $q_1 = (c, d, e)$ and $q_2 = (f, g, h)$ are **orthonormal column vectors in $\mathbf{R}^3$**. They span a subspace S.

(a) Find the $(1, 1)$ entry in the projection matrix P that projects each vector in $\mathbf{R}^3$ onto that subspace S.

Solution. Denote by Q the matrix with columns q_1 and q_2: $Q = \left[\begin{smallmatrix} c & f \\ d & g \\ e & h \end{smallmatrix}\right]$. The projection matrix $P = Q(Q^T Q)^{-1} Q^T$. As the column vectors are orthonormal, we know that $Q^T Q$ is the 2-by-2 identity matrix. Thus, $P = QQ^T$, and the first entry is $c^2 + f^2$.

(b) For this projection matrix P, describe 3 independent eigenvectors (vectors for which Px is a number λ times x). What are the 3 eigenvalues of P? What is its determinant?

Solution. The projection matrix P projects onto a 2d plane. That means its eigenvalues are $(1, 1, 0)$ and the determinant is 0. The eigenvector corresponding to the eigenvalue 0 is orthogonal to the projection plane, that is orthogonal to both vectors q_1 and q_2. The independent vectors corresponding to value 1 are any two independent vectors in the projection plane. We can choose q_1 and q_2 as such vectors.

(c) For some vectors v and w in $\mathbf{R}^3$ the Gram-Schmidt orthonormalization process (applied to v and w) will produce those particular vectors q_1 and q_2. **Describe** the vectors v and w that lead to this q_1 and q_2.

Solution. Vector v is on the same line as q_1 and in the same direction. Therefore, $v = aq_1$, where a is a positive number. The second vector w has to be in the same plane as q_1 and q_2, on the same side of the line drawn through q_1 as q_2 and has to be independent of v.

3. (34 points)

(a) If q_1, q_2, q_3 are orthonormal vectors in $\mathbf{R}^3$, what are the possible determinants of this matrix A with columns $2q_1$ and $3q_2$ and $5q_3$? **Why?**

$$A = \begin{bmatrix} 2q_1 & 3q_2 & 5q_3 \end{bmatrix}$$

Solution. The determinant of the matrix $Q = [q_1\ q_2\ q_3]$ has to be 1 or -1. This is because $Q^TQ = I$, which means that $\det Q^T \cdot \det Q = 1$, that is, $\det Q^2 = 1$. When we multiply a column by a number, the determinant is multiplied by the same number. Thus, the determinant of A is either 30 or -30.

(b) For a matrix A, suppose the cofactor C_{11} of the first entry a_{11} is **zero**. What information does that give about A^{-1}? Can this inverse exist?

Solution. The cofactor C_{11} being zero does not give us enough information to decide whether the inverse exists or not. For example, in the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ this cofactor is zero and the inverse does not exist, and in the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ this cofactor is zero and the inverse exists. If this inverse exists, then we know that the entry $(1, 1)$ in this inverse is zero.

(c) Find the 3 eigenvalues of this matrix A and find all of its eigenvectors. Why is the diagonalization $S^{-1}AS = \Lambda$ not possible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution. The eigenvalues of this matrix are $(2, 2, 2)$. But the rank of $A - 2I$ is 1. That means, you can only find two independent eigenvectors. When the number of independent eigenvectors is smaller than the size of the matrix, then the diagonalization is not possible because you cannot build the square matrix of eigenvectors S.