

Solutions

1. (36 points) Start with the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix}$$

- (a) Find a basis for the column space $\mathbf{C}(A)$.
- (b) Find a basis for the null space $\mathbf{N}(A)$.
- (c) Find a basis for the row space $\mathbf{C}(A^T)$.
- (d) Write the complete solution to $Ax = b$.

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Solution. We can notice that the second row is 2 times the first row. This means that rows are dependent and the rank of the matrix is less than 3. In addition, we see that not all the rows are multiples of the same row, that means the rank of the matrix is more than 1. Therefore, it must be equal to 2. It follows that $\dim \mathbf{C}(A) = \dim \mathbf{C}(A^T) = 2$. The dimension of the null space is $n - r = 4 - 2 = 2$.

We can also find the dimensions by calculating U . As the second row is proportional to the first one, we need to swap the second and the third row and have a $PA = LU$ decomposition:

$$U = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution (1a). Any two independent columns of A would form a basis. That means columns 1 and 3, or columns 2 and 3. The default answer is to pick the pivot columns: that is pick columns that are not linearly dependent on other columns to the left of them. In this case, these are columns 1 and 3: $(1, 2, 3)$ and $(2, 4, 7)$.

Solution (1b). The fact that the last column is zero means $(0, 0, 0, 1)$ is in the null space. The fact that the first column plus the second is zero means $(1, -1, 0, 0)$ is in the nullspace. Or, we can use U and the default basis for the null space is $(-1, 1, 0, 0)$ and $(0, 0, 0, 1)$.

Solution (1c). Similarly to 1a), we can pick any two independent rows of A as the basis. That means row 1 and row 3, or row 2 and row 3. Or, we can pick all the non-zero rows in the U matrix. The most common bases would be either $(1, -1, 2, 0)$ and $(3, -3, 7, 0)$, or $(1, -1, 2, 0)$ and $(0, 0, 1, 0)$.

Solution (1d). Using the the row swap and the elimination on b , the augmented matrix U is:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The particular solution is $(-1, 0, 1, 0)$ and the complete solution: $(-1, 0, 1, 0) + a(-1, 1, 0, 0) + b(0, 0, 0, 1)$.

2. (32 points)

- (a) Suppose the matrices A and B have the same column space. Give an example where A and B have different nullspaces — or say why this is impossible.

Solution. The simplest way to provide an example is to add dependent columns to matrix A . For example, matrices $[1]$ and $[1 \ 0]$ have the same column space and different null spaces.

- (b) Again A and B have the same column space. Give an example where A and B have different ranks r — or say why this is impossible.

Solution. The rank is the dimension of the column space. That means the rank is the same for both matrices.

- (c) CIRCLE TRUE or FALSE:

If B is a square matrix then $\mathbf{C}(B) = \mathbf{C}(B^T)$.

Solution. False. Consider $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then the column space of B is $(1, 0)^T$ and the column space of B^T is $(0, 1)^T$.

- (d) If the columns of a 5 by 3 matrix M are linearly independent and x in \mathbf{R}^3 is not the zero vector, then you know that Mx is _____.
I am looking for an answer that uses independence of columns and $x \neq 0$.

Solution. Mx is a non zero vector in \mathbf{R}^5 .

3. (32 points)

- (a) Find a 3 by 3 matrix A whose column space is the plane $x + y + z = 0$ in \mathbf{R}^3 . (This means: $\mathbf{C}(A)$ consists of all column vectors (x, y, z) with $x + y + z = 0$.)

Solution. The column space must have dimension 2. That means that any two independent vectors in the plane will do. For example,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

- (b) How do you know that a 3 by 3 matrix A with that column space is not invertible?

Solution. The plane is 2-dimensional, so the rank of the matrix is 2, which is less than the size of the matrix.

- (c) Does there exist a matrix B whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$? If so, construct B . If not, explain why not.

Solution. Such matrix does not exist. The dimensions of such a matrix must be 3 by 4 ($m = 3$ and $n = 4$). The dimension of the column space is 2, because the given vectors are independent. That means the dimension of the nullspace must be $4 - 2 = 2$. The null space cannot be spanned by 1 vector.

- (d) Is this set of matrices a vector space or not? All 3 by 3 matrices with the column vector $(1, 1, 1)$ in their column space. YES or NO with a reason.

Solution. Consider matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

They contain $(1, 1, 1)$ in their column space, but their sum does not:

$$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

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