① $\cos\left(x+\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos(x) - \sin\left(\frac{\pi}{3}\right)\sin(x)$

$$= \frac{1}{2}\cos(x) - \frac{\sqrt{3}}{2}\sin x$$

$A = \frac{1}{2}$ $\quad B = -\frac{\sqrt{3}}{2}$

$$P_a b = \frac{\langle a, b\rangle}{\langle a, a\rangle} a$$

$$\frac{\langle a, b\rangle}{\langle a, a\rangle} = \frac{1}{\pi}\langle \sin x, \cos\left(x+\frac{\pi}{3}\right)\rangle = \frac{1}{\pi}\left(\langle \sin x, \tfrac{1}{2}\cos x\rangle + \langle \sin x, -\tfrac{\sqrt{3}}{2}\sin(x)\rangle\right)$$

orthogonality

$$= \frac{1}{\pi}\left(0 - \frac{\sqrt{3}}{2}\int_0^{2\pi} \sin^2 x \, dx\right) = -\frac{\sqrt{3}}{2}\cdot\frac{\pi}{\pi} = -\frac{\sqrt{3}}{2}$$

Projection of $\cos\left(x+\frac{\pi}{3}\right)$ onto $\sin x$ is $-\frac{\sqrt{3}}{2}\sin x$.

(You could have obtained this directly from the above expression, but I worked it out to demonstrate that the process has not changed just because we switched to functions.)

② 10.2.1)

$$\|x\| = \left(\sum_i |x_i|^2\right)^{1/2} \qquad |x_i|^2 = a_i^2 + b_i^2$$

$$\|u\| = \left(|1+i|^2 + |1-i|^2 + |1+2i|^2\right)^{1/2} = (2+2+5)^{1/2} = 3$$

$$\|v\| = \left(|i|^2 + |i|^2 + |i|^2\right)^{1/2} = \sqrt{3}$$

$$u^H v = \begin{bmatrix} 1-i & 1+i & 1-2i \end{bmatrix} \begin{bmatrix} i \\ i \\ i \end{bmatrix} = 1+i-1+i+2+i = 2+3i$$

$$v^H u = \begin{bmatrix} -i & -i & -i \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \\ 1+2i \end{bmatrix} = 1-i-1-i+2-i = 2-3i$$

10.2.3)

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

$$Az = 0$$

$$\begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}} \begin{bmatrix} i & 1 & i \\ 0 & 2i & -1+i \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} -i & 0 \\ 0 & -\frac{1}{2}i \end{bmatrix}} \begin{bmatrix} 1 & -i & 1 \\ 0 & 1 & \frac{1}{2}(1+i) \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & \frac{1}{2}(1+i) \\ 0 & 1 & \frac{1}{2}(1+i) \end{bmatrix} \qquad s_1 = \begin{bmatrix} -\frac{1}{2}(1+i) \\ -\frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

or rescaled for convenience: $z = \begin{bmatrix} 1+i \\ 1+i \\ -2 \end{bmatrix}$

$z \in N(A)$

$$z^H A^H = \begin{bmatrix} 1-i & 1-i & -2 \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix} = \begin{bmatrix} -i-1+1-i+2i & 1-i-i-1+2i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix} = 0^H$$

10.2.8)

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} \qquad P^H = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$$

$$PP^H = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = I = P^H P \Rightarrow P \text{ is } \underline{\text{invertible}} \text{ w/ } P^H = P^{-1}$$

$$\Rightarrow P \text{ is } \underline{\text{unitary}}$$

$P \neq P^H \Rightarrow P$ is <u>not Hermitian</u>

$$P^2 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} = -iI$$

$$P^{100} = P^{99}P = (P^3)^{33}P = (-i)^{33}P = -iP$$

$$\det(P - \lambda I) = \det \begin{bmatrix} -\lambda & i & 0 \\ 0 & -\lambda & i \\ i & 0 & -\lambda \end{bmatrix} = -\lambda^3 - i = 0$$

$$\Rightarrow \lambda^3 = -i$$

$$\Rightarrow \lambda^3 = \frac{1}{i} \Rightarrow \lambda_k = \frac{\sqrt[3]{1}}{\sqrt[3]{i}} = \frac{1}{-i} e^{i2\pi \frac{k}{3}} = i e^{i2\pi \frac{k}{3}}$$

$$\sqrt[3]{1} = e^{i2\pi \frac{0}{3}}, e^{i2\pi \frac{1}{3}}, e^{i2\pi \frac{2}{3}}$$

$$\sqrt[3]{i} = -i$$

10.2.16) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\mathrm{Tr}\, Q = 2\cos\theta = \lambda_1 + \lambda_2$

$\det Q = \lambda_1 \lambda_2 = \cos^2\theta + \sin^2\theta$

$\Rightarrow \quad \lambda_1 = \cos\theta + i\sin\theta = e^{i\theta}$

$\lambda_2 = \cos\theta - i\sin\theta = e^{-i\theta}$

$(Q - \lambda_1 I)x_1 = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \frac{1}{\sqrt{2}}$

$(Q - \lambda_2 I)x_2 = \begin{bmatrix} \sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \frac{1}{\sqrt{2}}$

$$Q = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U^H}$$

(3) 7.1.3)

(a) Linear. $T(v) = Av$ for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) Linear. $T(v) = Av$ for $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(c) Linear $T(v) = Av$ for $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(d) Nonlinear. $T(u+v) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq T(u) + T(v) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(e) Linear. $T(v) = Av$ for $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$

(f) Nonlinear. $T(u+v) = (u_1+v_1)(u_2+v_2) = u_1u_2 + u_1v_2 + u_2v_1 + v_1v_2$

$\neq T(u) + T(v) = u_1u_2 + v_1v_2$

7.1.10) (a) $T(v_1, v_2) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ has $\begin{bmatrix} c \\ 0 \end{bmatrix} \in \text{Ker}(T)\ \forall c$

(b) $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \notin \text{Range}(T)\ \forall c \neq 0$

(c) $T(v_1, v_2) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1$ has $\begin{bmatrix} 0 \\ c \end{bmatrix} \in \text{Ker}(T)\ \forall c$