Math 18.06, Spring 2013
Problem Set #6
April 5, 2013

Problem 1 (§5.2, 4). *Identify all the nonzero terms in the big formula for the determinants of the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

For A, we must choose the 1 in the the entry $(2,3)$, since it's the only nonzero thing in the third column. This rules out all the other entries of the second row, and in the second column, we are again left with only a single choice, the 1 in entry $(3,2)$. There are then two ways to choose entries in the other two columns.

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & 1 & \boxed{1} & 1 \\ 1 & \boxed{1} & 0 & 1 \\ 1 & 0 & 0 & \boxed{1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & \boxed{1} \\ 0 & 1 & \boxed{1} & 1 \\ 1 & \boxed{1} & 0 & 1 \\ \boxed{1} & 0 & 0 & 1 \end{pmatrix}$$

The determinant of A is now

$$\begin{aligned} \det A &= (1)(1)(1)(1)\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (1)(1)(1)(1)\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= -1 + 1 = 0 \end{aligned}$$

Of course, the determinant has to be 0 – the first and last rows are the same.

For B, there are the same choices of nonzero entries:

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & 3 & \boxed{4} & 5 \\ 5 & \boxed{4} & 0 & 3 \\ 2 & 0 & 0 & \boxed{1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & \boxed{2} \\ 0 & 3 & \boxed{4} & 5 \\ 5 & \boxed{4} & 0 & 3 \\ \boxed{2} & 0 & 0 & 1 \end{pmatrix}.$$

The determinant is

$$\begin{aligned} \det B &= (1)(4)(4)(1)\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (2)(4)(4)(2)\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= -16 + 64 = 48. \end{aligned}$$

Problem 2 (§5.2, 9). *Show that* 4 *is the largest determinant for a* 3×3 *matrix of* 1*s and* -1*s.*

There are a couple ways to do this. Here's a tricky one. Suppose the matrix is

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

By the big formula,

$$\det A = aei + bfg + cdh - ahf - dbi - gec.$$

Since each of the entries is either $+1$ or -1, each of the terms here is either $+1$ or -1. That means that the sum of all of them is one of $-6, -4, -2, 0, 2, 4, 6$. To show that the maximum sum is 4, we just need to rule out its being 6.

The only way we could get 6 is if $aei = bfg = cdh = 1$ and $ahf = dbi = gec = -1$. Multiplying the first three together, this would say $abcdefghi = 1$ (that is to say, the number of -1s is even). Multiplying the last three together, we get $abcdefghi = -1$ (the number of -1s is odd). This is impossible! So there's no way to get 6.

Here's a slightly more systematic approach. We can expand by cofactors across the top row:

$$\det A = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Each of the 2×2 determinants here is -2, 0, or 2. If we can prove that at least one of them is 0, it follows that the total is at most 4.

Let's say a column is a "=-column" if the two bottom entries are the same, and a "≠-column" if they are different (i.e. one entry is $+1$ and one is -1). We have three columns, so there must be either two =-columns or two ≠-columns. If there are two =-columns, the cofactor corresponding to these two columns is 0. If there are two ≠-columns, the cofactor corresponding to these two columns is 0. So at least one of the cofactors has to be 0, and this means the maximum determinant is 4.

You might notice that these arguments won't extend very well if we try to answer the same questions for $n \times n$ matrices, instead of just 3×3. In fact, the answer isn't known – this is an old problem, called Hadamard's maximal determinant problem! You can Wiki it if you're curious. The maximum possible determinant is known for $n \leq 21$ but not past that.

Problem 3 (§5.2, 12). *Take*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

The cofactor matrix is

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then we compute $AC^T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Notice that since $\det A = 4$, $A^{-1} = C^T / \det A$ as expected.

Problem 4 (§5.2, 15). *We consider the tridiagonal matrices*

$$E_1 = \begin{vmatrix} 1 \end{vmatrix}, \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}, \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

a) To compute E_n, we expand by minors across the first row. There are two terms.

$$\begin{aligned} E_n = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} \times & \times & \times & \times & \times \\ \times & 1 & 1 & 0 & 0 \\ \times & 1 & 1 & 1 & 0 \\ \times & 0 & 1 & 1 & 1 \\ \times & 0 & 0 & 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} \times & \times & \times & \times & \times \\ 1 & \times & 1 & 0 & 0 \\ 0 & \times & 1 & 1 & 0 \\ 0 & \times & 1 & 1 & 1 \\ 0 & \times & 0 & 1 & 1 \end{vmatrix} \\ &= E_{n-1} - \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \end{aligned}$$

The first matrix in the cofactor expansion is just E_{n-1}, but the second is slightly different: it's missing a 1 in the $(2,1)$-entry. To compute its determinant, we do a cofactor expansion down the first column. This has only a single term, which is E_{n-2}:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot E_{n-2}$$

Putting this back in to our original expansion, we get $E_n = E_{n-1} - E_{n-2}$.

b) Since clearly $E_1 = 1$ and $E_2 = 0$, we use the above formula and obtain $E_3 = -1$, $E_4 = -1$, $E_5 = 0$, $E_6 = 1$, $E_7 = 1$, $E_8 = 0$.

c) These things are repeating $1, 0, -1, -1, 0, 1$ with a period of 6. This implies that $E_{100} = E_4 = -1$, since 100 is 4 more than a multiple of 6.

Problem 5 (§5.2, 23). a) *We have a* 4×4 *block matrix*

$$M = \left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} \right).$$

Why is $\det M = \det A \cdot \det D$*?*

There are four ways to pick nonzero entries in the big formula:

$$
\left(\begin{array}{cc|cc}
\boxed{a_{11}} & a_{12} & b_{11} & b_{12} \\
a_{21} & \boxed{a_{22}} & b_{21} & b_{22} \\
\hline
0 & 0 & \boxed{d_{11}} & d_{12} \\
0 & 0 & d_{21} & \boxed{d_{22}}
\end{array}\right), \qquad
\left(\begin{array}{cc|cc}
\boxed{a_{11}} & a_{12} & b_{11} & b_{12} \\
a_{21} & \boxed{a_{22}} & b_{21} & b_{22} \\
\hline
0 & 0 & d_{11} & \boxed{d_{12}} \\
0 & 0 & \boxed{d_{21}} & d_{22}
\end{array}\right)
$$
$$
\left(\begin{array}{cc|cc}
a_{11} & \boxed{a_{12}} & b_{11} & b_{12} \\
\boxed{a_{21}} & a_{22} & b_{21} & b_{22} \\
\hline
0 & 0 & \boxed{d_{11}} & d_{12} \\
0 & 0 & d_{21} & \boxed{d_{22}}
\end{array}\right), \qquad
\left(\begin{array}{cc|cc}
a_{11} & \boxed{a_{12}} & b_{11} & b_{12} \\
\boxed{a_{21}} & a_{22} & b_{21} & b_{22} \\
\hline
0 & 0 & d_{11} & \boxed{d_{12}} \\
0 & 0 & \boxed{d_{21}} & d_{22}
\end{array}\right)
$$

The determinant is now

$$
\begin{aligned}
\det M &= a_{11}a_{22}d_{11}d_{22} - a_{11}a_{22}d_{12}d_{21} - a_{12}a_{21}d_{11}d_{22} + a_{12}a_{21}d_{12}d_{21} \\
&= (a_{11}a_{22} - a_{12}a_{21})(d_{11}d_{22} - d_{12}d_{21}) \\
&= \det A \cdot \det D,
\end{aligned}
$$

as claimed.

The only way to come up with a nonzero thing in each row in each column is to do this for A and D separately, then combine them; the same trick works for block upper triangular matrices of other sizes too.

b) *Show that this doesn't work anymore if* $C \neq 0$*:*

$$
M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).
$$

It might not be true that $\det M = \det A \det D - \det B \det C$*.*

Here's an example:

$$
M = \left(\begin{array}{cc|cc}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right).
$$

Here M has $\det M = -1$ (it's a permutation matrix), but each of the submatrices has determinant 0.

c) The same example shows that $\det M = \det(AD - CB)$ doesn't work either. We have $AD = 0$ and $BC = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so $\det(AD - BC) = \det\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0$.

Problem 6 (§5.2, 34). *Let*

$$
A = \begin{pmatrix}
x & x & x & x & x \\
x & x & x & x & x \\
0 & 0 & 0 & x & x \\
0 & 0 & 0 & x & x \\
0 & 0 & 0 & x & x
\end{pmatrix}
$$

a) The last three rows have a span that is at most 2-dimensional; it follows that there is some linear dependence among them.

b) Every term in the big formula is 0. The reason is that in picking entries in rows 3, 4, and 5, at least one of them must be in a column other than 4 or 5, since we're only allowed to pick one entry per column. That means it must have a 0, and so the corresponding term in the big formula is 0.

Problem 7 (§5.3, 1). a) We use Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}, \det B_1 = -6, \qquad B_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \det B_2 = 3.$$

Then

$$x_1 = \frac{\det B_1}{\det A} = \frac{-6}{3} = -2,$$
$$x_2 = \frac{\det B_2}{\det A} = \frac{3}{3} = 1.$$

b) Now for a 3×3 $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \det B_1 = 3, \qquad B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \det B_2 = -2,$$
$$B_3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \det B_3 = 1.$$

Then

$$x_1 = \frac{\det B_1}{\det A} = \frac{3}{4},$$
$$x_2 = \frac{\det B_2}{\det A} = \frac{-2}{4} = -\frac{1}{2},$$
$$x_3 = \frac{\det B_3}{\det A} = \frac{1}{4}.$$

Problem 8 (§5.3, 12). *If all entries of A and A^{-1} are integers, prove that* $\det A = 1$ *or* -1.

If A has all entries integers, then $\det A$ is an integer: this is implied by the big formula, which says that the determinant is a sum of a bunch of terms, every single one of which is an integer. Now $\det A^{-1} = 1/\det A$. But if A^{-1} has all entries integers, then $\det A^{-1}$ is an integer too. This means that $\det A$ is an integer whose reciprocal is also an integer, and the only possibilities are 1 and -1.

Problem 9 (§5.3, 20). *Compute the determinant of*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

As pointed out in the question, this matrix has orthogonal rows. That means that $H^T H$ is diagonal, and computing its determinant is easy. In fact,

$$H^T H = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

This has determinant $4 \cdot 4 \cdot 4 \cdot 4 = 256$. But $\det(H^T H) = \det(H^T)\det(H) = \det(H)^2$, so $\det H = 16$.

Problem 10 (§5.3, 36). If (x, y, z), $(1, 1, 0)$, and $(1, 2, 1)$ all lie on a plane through the origin, then

$$\det \begin{pmatrix} x & 1 & 1 \\ y & 1 & 2 \\ z & 0 & 1 \end{pmatrix} = 0.$$

Expanding the determinant, this means $x - y + z = 0$, so that's the equation for the plane in question.

Problem 11 (§6.1, 9). *Suppose that $A\mathbf{x} = \lambda\mathbf{x}$, i.e. $\mathbf{x}$ is an eigenvector of A, with eigenvalue λ.*

a) We have $(A^2)\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2\mathbf{x}$. This means that $\mathbf{x}$ is also an eigenvector of A^2, and the corresponding eigenvalue is λ^2.

b) Starting with $A\mathbf{x} = \lambda\mathbf{x}$, we multiply by A^{-1} on the left of both sides, obtaining $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. This implies that $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$, i.e. that $\mathbf{x}$ is also an eigenvector of A^{-1}, with eigenvalue λ^{-1}.

c) Note that $(A+I)\mathbf{x} = A\mathbf{x} + I\mathbf{x} = \lambda\mathbf{x} + \mathbf{x} = (\lambda+1)\mathbf{x}$, which means that $\mathbf{x}$ is an eigenvector of $A + I$, with eigenvalue $\lambda + 1$.

Problem 12 (§6.1, 17). Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc).$$

Notice that the coefficient $a+d$ is the trace, while the coefficient $ad-bc$ is the determinant. The quadratic formula tells us the eigenvalues:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{(a+d) \pm \sqrt{(a-d)^2 + bc}}{2}$$

The sum of the two possibilities is $a+d$, while their product is $ad-bc$.

If $\lambda_1 = 3$ and $\lambda_2 = 4$, then the characteristic polynomial must be $(\lambda-3)(\lambda-4) = \lambda^2 - 7\lambda + 12$.

Problem 13 (§6.1, 19). *Suppose that B is a 3×3 matrix with eigenvalues 0, 1, 2.*

a) It follows that the rank of B is 2. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the eigenvectors. These are eigenvectors for different eigenvalues, so they are linearly independent and therefore a basis for $\mathbb{R}^3$.

If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is any vector in the space, then $B\mathbf{v} = c_1(0\mathbf{v}_1) + c_2(1\mathbf{v}_2) + c_3(2\mathbf{v}_3) = c_2\mathbf{v}_2 + 2c_3\mathbf{v}_3$. If c_2 or c_3 is nonzero, then $B\mathbf{v}$ is nonzero, since $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. So the nullspace of B is exactly the multiples of $\mathbf{v}_1$, a one-dimensional space. It follows that the rank is 2.

b) We have

$$\det(B^T B) = \det(B^T)\det(B) = \det(B^T) \cdot (0)(1)(2) = 0,$$

since the product of the eigenvalues is the determinant.

c) We don't have enough information to figure out the eigenvalues of $B^T B$ – knowing the eigenvalues of two matrices isn't enough to know the eigenvalues of the product.

d) To find the eigenvalues of $(B^2 + I)^{-1}$, we unwind things in three steps, using the rules for how changing a matrix changes the eigenvalues.

Matrix	Eigenvalues
B	0, 1, 2
B^2	0, 1, 4
$B^2 + I$	1, 2, 5
$(B^2 + I)^{-1}$	1, 1/2, 1/5

Problem 14 (§6.1, 30). *Suppose that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a+b = c+d$.

First we check that $(1,1)$ is an eigenvector.

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This means that $(1,1)$ is an eigenvector, with eigenvalue $a+b$ $(= c+d)$.

The sum of the two eigenvalues is equal to the trace, which is $a+d$. So the second eigenvalue must be $(a+d) - (a+b) = d-b$.

Problem 15. *We want to maximize the determinant of the symmetric matrix*

$$S = \begin{pmatrix} 4 & 2 & x \\ 2 & 4 & 2 \\ x & 2 & 4 \end{pmatrix}$$

a) By the big formula,

$$\begin{aligned} \det S &= 64 + 4x + 4x - 16 - 16 - 4x^2 = -4x^2 + 8x + 32 \\ &= -4(x^2 - 2x - 8) = -4((x-1)^2 - 9). \end{aligned}$$

This is maximized when $x = 1$.

b) To find the inverse, we can compute the cofactor matrix.

$$C = \begin{pmatrix} 12 & -6 & 0 \\ -6 & 15 & -6 \\ 0 & -6 & 12 \end{pmatrix}.$$

Then

$$S^{-1} = \frac{1}{\det S} C^T = \frac{1}{36} \begin{pmatrix} 12 & -6 & 0 \\ -6 & 15 & -6 \\ 0 & -6 & 12 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

c) The $(1,3)$-entry of the inverse was determined by the $(3,1)$-entry of the cofactor matrix, which is the determinant of the submatrix obtained by deleting the third row and the first column, i.e. $\left(\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right)$.

d) Here's another way to get the same answer. From the Lewis Carroll identity on the last problem set, we know that

$$4 \det S = (\det A)(\det D) - (\det B)(\det C),$$

where A, B, C, and D are the 3×3 submatrices corresponding to the top left, top right, bottom left, and bottom right respectively. Observe that $C = B^T$, so $\det B = \det C$ and thus

$$4 \det S = (\det A)(\det D) - (\det B)^2.$$

Since $(\det B)^2 \geq 0$, and since $\det A$ and $\det D$ don't depend on x, the largest this can possibly be is when $\det B = 0$. But $\det B = \det \left(\begin{smallmatrix} 2 & x \\ 4 & 2 \end{smallmatrix}\right) = 4 - 4x$, which is 0 exactly when $x = 1$. This agrees with our answer to part a) of this problem.