HOMEWORK SOLUTIONS

<u>6.2</u>

16

$$A_1 = \begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix}, \det(A_1 - \lambda I) = \det \begin{pmatrix} 0.6-\lambda & 0.9 \\ 0.4 & 0.1-\lambda \end{pmatrix}$$

$$= (0.6-\lambda)(0.1-\lambda) - (0.4)(0.9) = \lambda^2 - 0.7\lambda - 0.3 = (\lambda - 1)(\lambda + 0.3)$$

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 4v_1 = 9v_2$$

$$\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -0.3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = -v_2$$

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 9 \\ 4 \end{pmatrix}; \lambda_2 = -0.3, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix}$$

$$A_1^k = S_1 \Lambda_1^k S_1^{-1} \to \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 9 & 0 \\ 4 & 0 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 1 & 1 \\ 4 & -9 \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix} \text{(In the columns, we see the eigenvector } v_1 \text{ for the eigenvalue 1.)}$$

17

$$A_2 = \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix}, \det(A_2 - \lambda I) = \det \begin{pmatrix} 0.6-\lambda & 0.9 \\ 0.1 & 0.6-\lambda \end{pmatrix}$$

$$= (0.6-\lambda)^2 - (0.1)(0.9) = \lambda^2 - 1.2\lambda + 0.27 = (\lambda - 0.3)(\lambda - 0.9)$$

$$\Lambda_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.9 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = -3v_2$$

$$\begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.9 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = 3v_2$$

$$\lambda_1 = 0.3, v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \lambda_2 = 0.9, v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix}$$

$$A_2^{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{6} S_2 \Lambda_2^{10} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 4(0.9)^{10} \end{pmatrix} = \begin{pmatrix} 2(0.9)^{10} \\ \frac{2}{3}(0.9)^{10} \end{pmatrix}$$

$$A_2^{10} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{6} S_2 \Lambda_2^{10} \begin{pmatrix} -6 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -6(0.3)^{10} \\ 0 \end{pmatrix} = \begin{pmatrix} 3(0.3)^{10} \\ -(0.3)^{10} \end{pmatrix}$$

$$A_2^{10} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = A_2^{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + A_2^{10} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2(0.9)^{10} + 3(0.3)^{10} \\ \frac{2}{3}(0.9)^{10} - (0.3)^{10} \end{pmatrix}$$

20

$A = S\Lambda S^{-1}$, so $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = (\det \Lambda)(\det S)(\det S^{-1}) = \det \Lambda$

This works when A can be diagonalized.

21

$$ST = \begin{pmatrix} aq + bs & ar + bt \\ cq + ds & cr + dt \end{pmatrix}$$

$$TS = \begin{pmatrix} qa + rc & qb + rd \\ sa + tc & sb + td \end{pmatrix}$$

$$\operatorname{tr}(ST) = \operatorname{tr}(TS) = aq + bs + cr + dt$$

24 If $S^{-1}AS$ and $S^{-1}BS$ are diagonal, $S^{-1}(A+B)S$ is diagonal and $S^{-1}(cA)S$ are also diagonal. Thus, all such 4×4 matrices form a subspace.

If $S = I$, we are looking at all matrices A such that $S^{-1}AS = A$ is diagonal; so it is the subspace of all diagonal matrices (this has dimension 4).

26

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This matrix only has 1 eigenvectors; its column space and nullspace are both spanned by v (so they coincide).

35

$$B - \lambda I = \begin{pmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{pmatrix}$$

$$C - \lambda I = \begin{pmatrix} 5-\lambda & 7 \\ -3 & -4-\lambda \end{pmatrix}$$

$$\det(B - \lambda I) = (3-\lambda)(-3-\lambda) - 2(-5) = \lambda^2 + 1$$
$$\det(C - \lambda I) = (5-\lambda)(-4-\lambda) - 7(-3) = \lambda^2 - \lambda + 1$$

Let the eigenvalues of B (resp. C) be b_1, b_2 (c_1, c_2); then $b_1^4 = b_2^4 = 1, c_1^3 = -1, c_2^3 = -1$; so the eigenvalues of B^4 (resp. C^3) are $1, 1$ (resp. $-1, -1$). So $B^4 = I, C^3 = -I$.

6.3

1

$$A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$$
$$A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 + 3v_2 \\ v_2 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 4, \lambda_2 = 1$ (because A is upper triangular); the eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solutions:

$$u = c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} + c_2 e^t \\ -c_2 e^t \end{pmatrix}$$
$$t = 0 \Rightarrow c_1 + c_2 = 5, -c_2 = -2 \Rightarrow c_1 = 3, c_2 = 2, u = \begin{pmatrix} 3e^{4t} + 2e^t \\ -2e^t \end{pmatrix}$$

4

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 - 1 = \lambda(\lambda+2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -2$$

$$A \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w - v \\ v - w \end{pmatrix} = \begin{pmatrix} \lambda v \\ \lambda w \end{pmatrix}; \lambda = \lambda_1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda = \lambda_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{Solutions:} u = c_1 e^{0\cdot t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 e^{2t} \\ c_1 - c_2 e^{2t} \end{pmatrix}$$

$$t = 0 \Rightarrow \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 10 \end{pmatrix}; c_1 = 20, c_2 = 10 \Rightarrow \begin{pmatrix} 20 + 10e^{2t} \\ 20 - 10e^{2t} \end{pmatrix}$$

14 (a)

$$u_1 u_1' + u_2 u_2' + u_3 u_3' = u_1(cu_2 - bu_3) + u_2(au_3 - cu_1) + u_3(bu_1 - au_2) = 0$$

$$\Rightarrow \frac{d}{dt}(||u(t)||^2) = 2(u_1 u_1' + u_2 u_2' + u_3 u_3') = 0 \Rightarrow ||u(t)||^2 = ||u(0)||^2$$

(b)

$$\begin{aligned} Q^T &= (1 + At + \frac{A^2 t^2}{2!} + \cdots)^T = 1 + (A^T)t + \frac{(A^T)^2 t^2}{2!} + \cdots \\ &= 1 - At + \frac{A^2 t^2}{2!} - \cdots = e^{-At} \\ Q^T Q &= e^{-At} e^{At} = I \end{aligned}$$

16

$$\begin{aligned} \frac{du}{dt} &= Au - e^{ct} b; u = e^{ct} v \\ \frac{du}{dt} &= ce^{ct} v + e^{ct} \frac{dv}{dt} = e^{ct}(cv + \frac{dv}{dt}); Au - e^{ct} b = e^{ct}(Av - b) \\ \Rightarrow cv + \frac{dv}{dt} &= Av - b \\ \frac{dv}{dt} &= (A - cI)v - b \end{aligned}$$

If c is not an eigenvalue, $A - cI$ is invertible; using the result from $Q15$, $v = (A - cI)^{-1} b$; and

$$u = e^{ct}(A - cI)^{-1} b$$

is a particular solution.

23

$$A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}, A^2 = A$$

$$B = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}, B^2 = 0$$

$$C = A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C^2 = C$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + A(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots) = I + A(e-1) = \begin{pmatrix} e & 4(e-1) \\ 0 & 1 \end{pmatrix}$$

$$e^B = I + B + \frac{B}{2!} + \cdots = I + B = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, e^C = I + C + \frac{C}{2!} + \cdots = I + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^A e^B = \begin{pmatrix} e & -4 \\ 0 & 1 \end{pmatrix}, e^B e^A = \begin{pmatrix} e & 4e-8 \\ 0 & 1 \end{pmatrix}, e^{A+B} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$$

25

$$\begin{aligned} A^2 &= A \\ e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = I + A(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots) = I + A(e^t - 1) \\ &= \begin{pmatrix} e^t & 3e^t - 3 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

<u>8.3</u>

2

$$A = \begin{pmatrix} .9 & .15 \\ .1 & .85 \end{pmatrix}$$

$$\lambda_1 + \lambda_2 = \mathrm{tr}(A) = .9 + .85; \lambda_1 = 1 \Rightarrow \lambda_2 = 0.75$$

$$\begin{pmatrix} .9 & .15 \\ .1 & .85 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0.9v_1 + 0.15v_2 \\ 0.1v_1 + 0.85v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda = \lambda_1 \Rightarrow v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \lambda = \lambda_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}, S = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, S^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$A^k = S\Lambda^k S^{-1} \to S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \frac{1}{5} \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}$$

7 Challenge problem:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; Av = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ (1-a)v_1 + (1-b)v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

$$\lambda_1 + \lambda_2 = a + 1 - b; \lambda_1 = 1 \Rightarrow \lambda_2 = a - b$$

$$\lambda = \lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} b \\ 1-a \end{pmatrix}; \lambda = \lambda_2 = a - b \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & a-b \end{pmatrix}; S = \begin{pmatrix} b & 1 \\ 1-a & -1 \end{pmatrix}; S^{-1} = \frac{1}{a-1-b} \begin{pmatrix} -1 & -1 \\ a-1 & b \end{pmatrix}$$

$$A^k = S\Lambda^k S^{-1} \to \begin{pmatrix} b & 1 \\ 1-a & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{a-1-b} \begin{pmatrix} -1 & -1 \\ a-1 & b \end{pmatrix} = \begin{pmatrix} \frac{-b}{a-1-b} & \frac{-b}{a-1-b} \\ \frac{a-1}{a-1-b} & \frac{a-1}{a-1-b} \end{pmatrix}$$

$$\begin{pmatrix} \frac{-b}{a-1-b} & \frac{-b}{a-1-b} \\ \frac{a-1}{a-1-b} & \frac{a-1}{a-1-b} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix} \Leftrightarrow \frac{-b}{a-1-b} = 0.6 \Leftrightarrow 0.4b + 0.6a = 0.6$$

For $a = 0.8, b = 0.3$ this is true.

9

$$A^2 = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}^2 = \begin{pmatrix} a^2 + b - ab & ab + b - b^2 \\ -a^2 + 1 + ab - b & -ab + 1 + b^2 - b \end{pmatrix}$$

All the entries of A^2 are positive (since all entries of A are positive); and the columns of this matrix also add up to 1, so A^2 is Markov.

14

$$(I + A)(I - A + A^2 - A^3 + \cdots) = (I - A + A^2 - A^3 + \cdots) + (A - A^2 + A^3 + \cdots) = I$$

$$A^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; A^{2k} = \begin{pmatrix} \frac{1}{2^k} & 0 \\ 0 & \frac{1}{2^k} \end{pmatrix}, A^{2k+1} = A \cdot A^{2k} = \begin{pmatrix} 0 & \frac{1}{2^{k+1}} \\ \frac{1}{2^k} & 0 \end{pmatrix}$$

$$I + A + A^2 + \cdots = \begin{pmatrix} 1 + \frac{1}{2} + \frac{1}{4} + \cdots & \frac{1}{2} + \frac{1}{4} + \cdots \\ 1 + \frac{1}{2} + \frac{1}{4} + \cdots & 1 + \frac{1}{2} + \frac{1}{4} + \cdots \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$