Math 18.06, Spring 2013
Problem Set #Exam 3
May 14, 2013

**Problem 1.** *In all of this problem, the 3 by 3 matrix $A$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with independent eigenvectors $x_1, x_2, x_3$.*

a) *What are the trace of $A$ and the determinant of $A$?*

The trace of $A$ is $\lambda_1 + \lambda_2 + \lambda_3$ and the determinant is $\lambda_1\lambda_2\lambda_3$.

b) *Suppose: $\lambda_1 = \lambda_2$. Choose the true statement from 1, 2, 3:*

1. *$A$ can be diagonalized.*
2. *$A$ can not be diagonalized.*
3. *I need more information to decide.*

(1) is the correct option, because we know that there exists a full set of independent eigenvectors.

c) *From the eigenvalues and eigenvectors, how could you find the matrix $A$? Give a formula for $A$ and explain each part carefully.*

We can recover $A$ using $A = S\Lambda S^{-1}$, where $S$ is a matrix whose columns are $x_1, x_2, x_3$, and $\Lambda$ is a diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \lambda_3$.

d) *Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $x_1 = (1, 1, 1)$ and $x_2 = (1, -2, 1)$. Choose $\lambda_3$ and $x_3$ so that $A$ is symmetric positive semidefinite but not positive definite.*

If we want $A$ to be symmetric, the third eigenvector $x_3$ had better be orthogonal to the other two. The quick way to find a vector orthogonal to two given ones in $\mathbb{R}^3$ is via cross product: $x_3 = x_1 \times x_2 = (3, 0, -3)$.

Alternately, you can use elimination: $x_3$ should be in the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

You could also just notice the first and last entries match and guess the answer from that. Either way, $x_3$ should be a multiple of $(1, 0, -1)$.

As for the eigenvalue, to get a matrix that's positive semidefinite but not positive definite, we need to use $\lambda_3 = 0$.

It doesn't actually ask you to compute $A$, but here's one that works:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 & 3 \\ -2 & 8 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

**Problem 2.** *Suppose $A$ has eigenvalues* 1, 1/3, 1/2 *and its eigenvectors are the columns of $S$:*

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{with} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

a) *What are the eigenvalues and eigenvectors of $A^{-1}$?*

The eigenvectors of $A^{-1}$ are the same as those of $A$. Its eigenvalues are the inverses of those of $A$: 1, 3, and 2.

b) *What is the general solution (with 3 arbitrary constants $c_1, c_2, c_3$) to the differential equation $du/dt = Au$? Not enough to write $e^{At}$. Use the $c$'s.*

The general solution is

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3 \\ &= c_1 e^t \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 e^{t/3} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t/2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

c) *Start with the vector $u = (1,4,3)$ from adding up the three eigenvectors: $u = x_1 + x_2 + x_3$. Think about the vector $v = A^k u$ for VERY large powers $k$. What is the limit of $v$ as $k \to \infty$?*

We have

$$A^k u = A^k (x_1 + x_2 + x_3) = \lambda_1^k x_1 + \lambda_2^k x_2 + \lambda_3^k x_3 = x_1 + \left(\frac{1}{3}\right)^k x_2 + \left(\frac{1}{2}\right)^k x_3.$$

When $k$ is very large, the two rightmost terms both go to 0, while the first one is an unchanging $x_1$. The limit $v$ is therefore equal to $x_1$.

**Problem 3.** a) *For a really large number $N$, will this matrix be positive definite? Show why or why not.*

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

The easiest test to use here is going to be to check whether the upper-left determinants are positive.

$1 \times 1$: This is 2, which is always greater than 0.

$2 \times 2$: This is $2N - 16$, which is greater than 0 if $N$ is really large (in particular if $N > 8$).

$3 \times 3$: Use the method of your choice to compute the determinant of $A$, in terms of $N$. By the (not so) big formula, it's

$$\det A = 8N + 12 + 12 - 2 - 64 - 9N = -42 - N.$$

This is going to be very negative if $N$ is really large. So the matrix will not be positive definite.

b)

*Suppose:* *$A$ is positive definite symmetric*
*$Q$ is orthogonal (same size as $A$)*
*$B$ is $Q^T AQ = Q^{-1}AQ$.*

*Show that:* *$B$ is also symmetric.*
*$B$ is also positive definite.*

First we show that $B$ is symmetric. This means we need to check $B^T = B$. Using what we're told,

$$B^T = (Q^T AQ)^T = Q^T A^T (Q^T)^T = Q^T AQ = B,$$

Note that in the next-to-last step we used the fact that $A$ itself is symmetric ($A^T = A$).

For positive definiteness, one way is to use the energy test. If $x$ is any nonzero vector, then

$$x^T Bx = x^T (Q^T AQ)x = (Qx)^T A(Qx) = y^T Ay,$$

where $y = Qx$. We know that $y$ is nonzero, because $Q$ is orthogonal and therefore has no nullspace.

Another approach is via eigenvalues. We know that $B = Q^{-1}AQ$, so $B$ is similar to $A$. That means that they have the same eigenvalues. Since $A$ is positive definite, its eigenvalues are all positive, so those of $B$ are as well.

A third approach: $A$ is positive definite, so $A = R^T R$ for some $R$ with independent columns. Then $B = Q^T R^T RQ = (RQ)^T (RQ)$. $RQ$ is a matrix with independent columns, since $Q$ is orthogonal. So $B$ is positive definite.

c) *If the SVD of $A$ is $U\Sigma V^T$, how do you find the orthogonal $V$ and the diagonal $\Sigma$ from the matrix $A$?*

First compute the matrix $A^T A$. Find the eigenvalues and eigenvectors. The first $r$ columns of $V$ should be length 1 eigenvectors of $A^T A$, corresponding to nonzero eigenvalues, arranged in order of decreasing eigenvalue. (A small complication: if there is a repeated eigenvalue, make sure to pick orthogonal eigenvectors for that eigenvalue). The diagonal entries of $\Sigma$ should be square roots of the eigenvalues of $A^T A$, again in decreasing order.

The remaining columns of $V$ should be an orthonormal basis for the nullspace of $A^T A$ (which is the same thing as the nullspace of $A$). This will give enough columns for $V$ to be a square matrix. The other diagonal entries of $\Sigma$ should be 0.