18.06 Spring 2013: PSet 5 Solutions

Solutions worked out on attached pages. Answers below, for quick reference.

1. (16) unknown $= \frac{9}{10}$

   (17) $\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$, $b = 9 + 4t$ (see attached for drawing)

2. (26) $\begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}$, $b = 2 - \frac{3}{2}x - \frac{3}{2}y$, $b = 2 = \frac{1}{4}(0+1+3+4)$

3. (8) Closest combination is sum of projection of b onto q_1 and q_2: $\hat{b} = (q_1^T b)q_1 + (q_2^T b)q_2$

   (10) (*a*) $0 = q_1^T(c_1q_1 + c_2q_2 + c_3q_3) = c_1q_1^Tq_1 + c_2q_1^Tq_2 + c_3q_1^Tq_3 = c_1$, and similar for c_2 and c_3

   (*b*) $0 = Q^T 0 = Q^T Qx = Ix = x$

4. (11) (*a*) $q_1 = \dfrac{1}{10}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$, $q_2 = \dfrac{1}{10}\begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$

   (*b*) $QQ^Tv = \dfrac{1}{100}\begin{bmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.50 \\ -0.18 \\ -0.24 \\ 0.40 \\ 0.00 \end{bmatrix}$

   (18) $A = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$

5. (24) (*a*) S is nullspace of $A = [\,1 \;\; 1 \;\; 1 \; -1]$, so special solutions s_1, s_2, s_3 form a basis for S.

   $$s_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\; s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\; s_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   (*b*) $S^\perp = C(A^T) \Rightarrow$ basis is $a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

   (*c*) $b = P_S b + P_{S^\perp} b = b_1 + b_2$

   $$b_2 = P_{S^\perp} b = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

   $$b_1 = P_S b = b - b_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

6. (1) (a) $\det(2A) = 8$
   (b) $\det(-A) = 1/2$
   (c) $\det(A^2) = 1/4$
   (d) $\det(A^{-1}) = 2$

   (5) $J_n \to I_n$ takes $n/2$ permutations if n is even and $(n-1)/2$ permuations if n is odd. This implies $J_{101} \to I_{101}$ takes $n = 50$ permutations.
   $\det(J_{101}) = (-1)^{50}\det(I_{101}) = 1$

7. (12) The rule for determinant of a scalar times a matrix was not applied correctly, $(1/(ad - bc))^2$ should appear.

   (15) (a) Determinant of first matrix is 0, by subtracting row 2 from row 3 and row 1 from row 2, then observing that the new row 2 and row 3 are the same.
   (b) Determinant of the second matrix is $1 - 2t^2 + t^4 = (1 - t^2)^2$ by subtracting t(row 1) from row 2 and t^2(row 1) from row 3, and then subtracting t(new row 2) from (new row 3) to get a triangular matrix.

8. (34) B is 1 permuation and 3 row sums of A, so $\det(B) = -6$.

9. $P_0(x) = \sqrt{2}/2$,
   $P_1(x) = \sqrt{3/2x}$,
   $P_2(x) = \sqrt{45/8}(x^2 - 1/3)$

   Also OK if 'standardized' to $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (1/2)(3x^2 - 1)$

10. See attached for development.

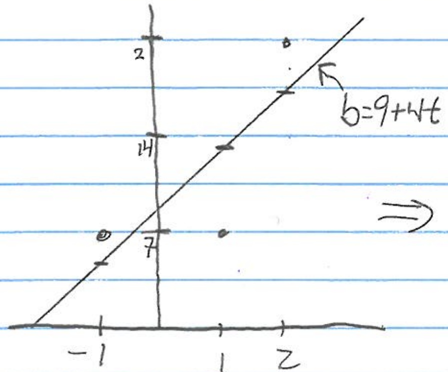
① ⑯ $\hat{x}_{10} = \frac{1}{10} b_{10} + C \hat{x}_9 = \frac{1}{10} \sum_{i=1}^{10} b_i$

$\hat{x}_9 = \frac{1}{9} \sum_{i=1}^{9} b_i \Rightarrow C \hat{x}_9 = C \frac{1}{9} \sum_{i=1}^{9} b_i = \frac{1}{10} \sum_{i=1}^{9} b_i \Rightarrow \boxed{C = \frac{9}{10}}$

⑰ $b = C + Dt$

$$Ax \simeq b \iff \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b \iff \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$



$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}} \quad b = 9 + 4t$$

② ㉖ $b = C + Dx + Ey$

$$Ax \simeq b \iff \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b \iff \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}}$$

$b = \hat{C} + \hat{D}x + \hat{E}y = 2 - \frac{3}{2}x - \frac{3}{2}y$

At $(x, y) = (0, 0)$, $b = 2 = \frac{1}{4}(0 + 1 + 3 + 4)$ = average of b's

3 8 $q_1, q_2 \in \mathbb{R}^5$ and orthonormal. The closest combination of q_1 and q_2 to b is the projection of b.

$$\hat{b} = (q_1^T b) q_1 + (q_2^T b) q_2$$

10 a $c_1 q_1 + c_2 q_2 + c_3 q_3 = 0 \Rightarrow q_1^T (c_1 q_1 + c_2 q_2 + c_3 q_3) = q_1^T 0 = 0$

$$\Rightarrow c_1 \overbrace{q_1^T q_1}^{1} + c_2 \overbrace{q_1^T q_2}^{0} + c_3 \overbrace{q_1^T q_3}^{0} = 0$$

$\Rightarrow c_1 = 0$ (Similar for $c_2 = 0, c_3 = 0$)

b $Qx = 0 \Rightarrow \overbrace{Q^T Q}^{I} x = Q^T 0 = 0$

$\Rightarrow Ix = 0 \Rightarrow x = 0$

4 11 a $a = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$ $b = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}$ $\boxed{q_1 = \frac{a}{\|a\|} = \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}}$

$q_1^T b = \frac{1}{10} [1\ 3\ 4\ 5\ 7] \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} = \frac{1}{10} 100 = 10$ $\tilde{q}_2 = b - (q_1^T b) q_1 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - 10 \cdot \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$

$$\boxed{q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{10} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}}$$

(4) (11) (b) Closest to $V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is projection of v onto $Q = [q_1 \; q_2]$

$$Q(Q^TQ)^{-1}Q^TV = QQ^TV$$

$$\frac{1}{100}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ -7 & 3 & 4 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -7 \end{bmatrix} = \boxed{\frac{1}{100}\begin{bmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.18 \\ -0.24 \\ 0.4 \\ 0 \end{bmatrix}}$$

(18) $A = a = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $\quad B = b - P_A b = b - a\underbrace{(a^Ta)^{-1}}_{1/2}\underbrace{a^Tb}_{-1} = b + \frac{1}{2}a = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$

$$C = c - P_A c - P_B c = C - a\overbrace{(a^Ta)^{-1}}^{1/2}\overbrace{a^Tc}^{0} - B\overbrace{(B^TB)^{-1}}^{2/3}\overbrace{B^Tc}^{-1}$$

$$= C + \frac{2}{3}B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}}$$

5 24 a) Subspace S of $\mathbb{R}^4$ is solutions of $Ax=0$ for $A = [1\ 1\ 1\ -1]$ or Nullspace of A.

$Ax=0$ has special solutions $s_1 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$ $s_2 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$ $s_3 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

$\boxed{s_1, s_2, s_3 \text{ form a basis for } S = N(A)}$

b) $\mathbb{R}^4 = N(A) + C(A^T)$, $S = N(A) \Rightarrow S^\perp = C(A^T)$

$$\boxed{\text{a basis for } S^\perp = C(A^T) \text{ is } a = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}}$$

c) $b = b_1 + b_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\quad b = P_S b + P_{S^\perp} b = b_1 + b_2$

$$P_{S^\perp} b = a(a^Ta)^{-1}a^Tb = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\Bigg(\overbrace{[1\ 1\ 1\ -1]\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}}^{1/4}\Bigg)^{-1}\overbrace{[1\ 1\ 1\ -1]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}^{2} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix} = b_2$$

$$P_S b = b - P_{S^\perp} b = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix} = b_1$$

$$\boxed{b_1 = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix} \qquad b_2 = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}}$$

⑥ ① $A \in \mathbb{R}^{4\times 4}$ $\det(A) = \frac{1}{2}$

ⓐ $\det(2A) = \det(2IA) = \det(2I)\det(A) = 2^4 \frac{1}{2} = 8$

ⓑ $\det(-A) = \det(-IA) = \det(-I)\det(A) = (-1)^4 \frac{1}{2} = \frac{1}{2}$

ⓒ $\det(A^2) = \det(A)\det(A) = \frac{1}{2}^2 = \frac{1}{4}$

ⓓ $\det(A^{-1}) = \dfrac{1}{\det(A)} = 2$

⑤ $J_5 \to I_5$: 2 permutations $\det(J_5) = (-1)^2 \det(I_5) = 1$

$J_6 \to I_6$: 3 permutations $\det(J_6) = (-1)^3 \det(I_6) = -1$

$J_7 \to I_7$: 3 permutations $\det(J_7) = (-1)^3 \det(I_7) = -1$

$J_n \to I_n$: $\begin{cases} n/2 & \text{permutations } n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$

$J_{101} \to I_{101}$: 50 permutations $\det(J_{101}) = (-1)^{50} \det(I_{50}) = 1$

(7) (12) $\det(A^{-1}) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \det\left(\frac{1}{ad-bc} I\right) \det\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

$$= \left(\frac{1}{ad-bc}\right)^2 (da - (-b)(-c)) = \frac{1}{ad-bc}$$

Rule for scalar times a matrix not applied correctly.

(15) (a) $A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = 0$$

because subtracting multiples of one row from another does not change the determinant and $\det(A) = 0$ if two rows are equal.

(b) $B = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}\right) = \underset{\substack{\text{row2} - t\,\text{row1} \\ \text{row3} - t^2\text{row1}}}{\det\left(\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix}\right)} = \underset{\text{row 3} - t\,\text{row 2}}{\det\left(\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{bmatrix}\right)}$$

$$\det(B) = 1 \cdot (1-t^2)(1-t^2) = 1 - 2t^2 + t^4$$

(8) (34) $A = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix}$ $B = \begin{bmatrix} - r_3 + r_2 + r_1 - \\ - r_2 + r_1 - \\ - r_1 - \end{bmatrix}$

$\det(A) = 6$

$B = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = SPA$

$\det(B) = \det(SPA) = \det(S)\det(P)\det(A) = 1 \cdot (-1) \cdot 6 \boxed{= -6}$

(9) $f_0(x) = 1 \qquad f_1(x) = x \qquad f_2(x) = x^2 \qquad \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$\tilde{P}_0(x) = 1 \qquad \|f\| = \langle f, f \rangle^{1/2}$

$$\tilde{P}_1(x) = f_1(x) - \frac{\langle \tilde{P}_0, f_1 \rangle}{\langle \tilde{P}_0, \tilde{P}_0 \rangle} \tilde{P}_0(x) = f_1(x) = x$$

$$\langle \tilde{P}_0, f_1 \rangle = \int_{-1}^{1} \tilde{P}_0(x) f_1(x)\,dx = \int_{-1}^{1} 1x\,dx = \left[\tfrac{1}{2}x^2\right]_{-1}^{1} = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$\tilde{P}_2(x) = f_2(x) - \frac{\langle \tilde{P}_0, f_2 \rangle}{\langle \tilde{P}_0, \tilde{P}_0 \rangle} \tilde{P}_0(x) - \frac{\langle \tilde{P}_1, f_2 \rangle}{\langle \tilde{P}_1, \tilde{P}_1 \rangle} \tilde{P}_1(x) = x^2 - \frac{\frac{2}{3}}{2} = x^2 - \frac{1}{3}$$

$$\langle \tilde{P}_0, f_2 \rangle = \int_{-1}^{1} 1x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{2}{3} \qquad \langle \tilde{P}_1, f_2 \rangle = \int_{-1}^{1} x \cdot x^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

$$\langle \tilde{P}_0, \tilde{P}_0 \rangle = \int_{-1}^{1} 1\,dx = 2 \qquad \langle \tilde{P}_1, \tilde{P}_1 \rangle = \int_{-1}^{1} x \cdot x = \tfrac{2}{3}$$

Now normalize...

$$\boxed{\begin{aligned} P_0 &= \tilde{P}_0 / \|\tilde{P}_0\| = 1/\sqrt{2} = \sqrt{2} \cdot \tfrac{1}{2} \\ P_1 &= \tilde{P}_1 / \|\tilde{P}_1\| = \sqrt{\tfrac{3}{2}}\, x \\ P_2 &= \tilde{P}_2 / \|\tilde{P}_2\| = \sqrt{\tfrac{45}{8}} \left(x^2 - \tfrac{1}{3}\right) \end{aligned}}$$

$$\langle \tilde{P}_2, \tilde{P}_2 \rangle = \int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right)\left(x^2 - \tfrac{1}{3}\right) dx = \left[\tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x\right]_{-1}^{1} = \tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9} + \tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9} =$$

$$= \tfrac{2}{5} - \tfrac{2}{9} = \tfrac{8}{45}$$

(10)

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad A = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \quad B = \begin{bmatrix} b & c \\ e & f \end{bmatrix} \quad C = \begin{bmatrix} d & e \\ g & h \end{bmatrix} \quad D = \begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

$\det A = ae - bd \quad \det B = bf - ce \quad \det C = dh - eg \quad \det D = ei - fh$

$$(\det A)(\det D) = (ae - bd)(ei - fh) = aeei - aefh - bdei + bdfh$$
$$= e(aei - afh - bdi) + bdfh$$

$$(\det B)(\det C) = (bf - ce)(dh - eg) = bfdh - bfeg - cedh + ceeg$$
$$= bfdh + e(-bfg - cdh + ceg)$$

$$(\det A)(\det D) - (\det B)(\det C) = e(aei - afh - bdi) + bdfh$$
$$- bdfh + e(bfg + cdh - ceg)$$

$$= e(aei + bfg + cdh - ceg - afh - bdi)$$

$$= e \det M \qquad (\text{or } m_{22}(\det M))$$