18.06 Spring 2013 – Problem Set 9 Solutions

1. 6.6 #4: Since A has two distinct eigenvalues, it must have 2 linearly independent eigenvectors. Since A is 2 by 2, this means that A is diagonalizable: $A = S\Lambda S^{-1}$ where $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It follows that if B is 2 by 2 and has the same eigenvalues, then $B = T\Lambda T^{-1}$ where T is the matrix of eigenvectors for B, and hence $B = TS^{-1}A(TS^{-1})^{-1}$ is similar to A.

2. 6.6 #6: Two 2 x 2 matrices with distinct eigenvalues are similar if and only if they have the same eigenvalues. This gives the following families:

$$\begin{aligned}
1, 0 &: \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\
1, -1 &: \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
2, 0 &: \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
\frac{1 \pm \sqrt{5}}{2} &: \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

This leaves 6 matrices with a repeated eigenvalue. The zero matrix and the identity matrix are each in their own family. The remaining matrices have a repeated eigenvalue but only one eigenvector (up to scaling). Using the Jordan form, we see these are in the same family if and only if they have the same (repeated) eigenvalue:

$$\begin{aligned}
1, 1 &: \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
0, 0 &: \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

3. 6.6 #17:

   (a) False: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (see previous question).

   (b) True: If B is singular it has 0 as an eigenvalue, and so if A is similar to B it must also have 0 as an eigenvalue and hence cannot be invertible.

   (c) False: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $-A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, since they both have a repeated eigenvalue of 0 with only one eigenvector, and therefore have the same Jordan form.

   (d) True: The eigenvalues of $A + I$ are equal to $\lambda_i + 1$ where λ_i are the eigenvalues of A. So $A + I$ cannot have the same eigenvalues as A and hence is not similar to A.

4. 6.6 #18: $AB = B^{-1}(BA)B$ so AB is similar to BA.

5. 6.6 #20:

   (a) If $B = M^{-1}AM$ then $B^2 = M^{-1}AMM^{-1}AM = M^{-1}A^2M$ so A^2 and B^2 are similar.

   (b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly A and B are not similar, but $A^2 = B^2$.

   (c) Both have the same distinct eigenvalues: 3 and 4.

   (d) Both have the same repeated eigenvalue of 3. However the first matrix has two linearly independent eigenvectors while the second does not, hence they are not similar.

   (e) $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

6. 6.7 #3: If A is rank 1, then so is A^TA and so its only non-zero eigenvalue is σ_1^2. The trace of A^TA is the sum of all of its eigenvalues, hence is equal to σ_1^2. Since the trace of A^TA is the sum of a_{ij}^2, it follows that σ_1^2 is the sum of all a_{ij}^2.

7. 6.7 #6:

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   has eigenvalues $3, 1, 0$ and unit eigenvectors $(1,2,1)/a, (1,0,-1)/b$ and $(1,-1,1)/c$ where $a = \sqrt{6}$, $b = \sqrt{2}$ and $c = \sqrt{3}$.

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

   has eigenvalues $3, 1$ and eigenvectors $(1,1)/d$ and $(1,-1)/e$ where $d = \sqrt{2}$, $e = \sqrt{2}$. The SVD decomposition for A is then:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

8. 6.7 #7: The best rank one approximation to A is $U\Sigma'V^T$ where Σ' is obtained from Σ by keeping the largest singular value and replacing the rest by zeroes. Another way of writing this is $\sigma_1 u_1 v_1^T$. Here this is given by:

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

9. 6.7 #14: We are looking for the closest rank one approximation to A. As above, this obtained by setting $\sigma_2 = 0$ in the singular value decomposition (i.e. is given by $\sigma_1 u_1 v_1^T$).

10. Σ and Λ agree if and only if A is a positive semi-definite matrix, i.e. A is symmetric and has non-negative eigenvalues. One direction is clear: if A is positive semidefinite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ then $A^T A = A^2$ has eigenvalues $\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2$, and hence the singular values are equal to the eigenvalues $\lambda_1, \ldots, \lambda_n$.

The other direction is more difficult. Suppose A has singular value decomposition $A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$, and suppose that the eigenvalues of A are equal to the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. We will show that A is symmetric (hence positive semi-definite since the eigenvalues are non-negative) by induction on the rank r. The case $r = 0$ is trivial since then $A = A^T = 0$. So suppose $r \geq 1$. For simplicity assume σ_1 is strictly greater than σ_2. Let x be a unit-length eigenvector for A corresponding to the eigenvalue σ_1, we claim that $x = \pm v_1$. To prove this write x in terms of the v's: $x_1 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + v_n \alpha_n$. Since $x^T x = 1$ we have $\alpha_1^2 + \cdots + \alpha_n^2 = 1$. We then have

$$\sigma_1^2 = \sigma_1^2 x^T x = (Ax)^T Ax = x^T (A^T Ax) = \sum_{i=1}^n \alpha_i v_i^T \left(\sum_{j=1}^n \alpha_j \sigma_j^2 v_j \right) = \sum_{i=1}^n \alpha_i^2 \sigma_i^2.$$

Now if $x \neq \pm v_1$ we must have $\alpha_i \neq 0$ for some $i > 0$, and since $\sigma_i^2 < \sigma_1^2$ for $i \neq 0$ we have

$$\sigma_1^2 = \sum_{i=1}^n \alpha_i^2 \sigma_i^2 < \sum_{i=1}^n \alpha_i^2 \sigma_1^2 = \sigma_1^2 \sum_{i=1}^n \alpha_i^2 = \sigma_1^2,$$

which is a contradiction. So we must have $x = \pm v_1$. Since $u_1 = A v_1 / \sigma_1$ it follows that $u_1 = v_1$ so $A = \sigma_1 v_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$. Now $B = A - \sigma_1 v_1 v_1^T$ will have eigenvalues and singular values equal to $\sigma_2, \cdots, \sigma_r, 0$. Since the rank of B is $r - 1$, by induction it must be symmetric and therefore $A = B + \sigma_1 v_1 v_1^T$ is symmetric as well.