Math 18.06, Spring 2013
Problem Set #1
February 14, 2013

Problem 1 (§2.2, 32). *We're looking at a singular system of* 100 *equations* $A\mathbf{x} = \mathbf{0}$.

a) That the system is singular means that some (non-trivial) linear combination of the 100 rows is the zero vector, i.e. a row of 100 zeroes.

b) Similarly, some linear combination of the 100 columns in a column of 100 zeroes.

c) One way to come up with such a system is the following. Fill in the first 99 rows of A with whatever you want, and then set the last row to be the sum of the first 100. We know this is singular because some linear combination of the rows is $\mathbf{0}$: $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_{99} - \mathbf{r}_{100} = \mathbf{0}$.

d) First let's try to understand this example using the row picture. $A\mathbf{x} = \mathbf{0}$ means that $\mathbf{x} \cdot \mathbf{r}_i = 0$ for every row $\mathbf{r}_i$; for each i, this determines a certain hyperplane that must contain $\mathbf{x}$. Having a nonzero solution means that these hyperplanes all contain a given line, the set of multiples of $\mathbf{x}$ (note that this is unusual; n hyperplanes chosen at random would intersect at only the point $\mathbf{0}$, as is clear when $n = 3$).

For the column picture, we imagine 100 different vectors, but they're all contained in a single hyperplane.

Problem 2 (§2.3, 3, 7). #3) The row reduction here runs

$$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow{\mathbf{r}_2 -= 4\mathbf{r}_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow{\mathbf{r}_3 += 2\mathbf{r}_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{\mathbf{r}_3 -= 2\mathbf{r}_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

The corresponding matrices are:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Then

$$M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}.$$

If you want to double-check this, you can compute

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

#7) E_{31} subtracts 7 times row 1 from row 3. To reverse it, we need to add 7 times row 1 to row 3. The matrix for this move is $E^{-1} = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{smallmatrix}\right]$, which is thus the inverse of of $E = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{smallmatrix}\right]$. Thinking about what this matrices do, it's clear that $EE^{-1} = E^{-1}E = I_3$.

Problem 3 (§2.4, 11). a) We know that $IA = A$ for any A, so $(4I)A = 4A$ for all A; we want to take $B = 4I$.

b) In order that $BA = 4B$ for all A, it must work in the particular case that $A = I$, so $B = 4B$. The only matrix satisfying this is $B = 0$.

c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

d) $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Problem 4 (§2.4, 33). We know that

$$A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This implies that

$$A(3\mathbf{x}_1 + 5\mathbf{x}_2 + 8\mathbf{x}_3) = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix},$$

which is what we want. So we should use $\mathbf{x} = 3\mathbf{x}_1 + 5\mathbf{x}_2 + 8\mathbf{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$.

To find the inverse, stick the three equations above into a matrix form, using the rule $A[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = [A\mathbf{x}_1\ A\mathbf{x}_2; A\mathbf{r}_3]$, which gives

$$A\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$, which you can compute by elimination. The answer is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Problem 5 (§2.4, 36). Doing a multiplication of $(m \times n)(n \times p)$ requires n multiplications for each entry of the product, a total of mnp. Working this out for the two possible orders of multiplying, $(AB)C$ requires $mnp + mpq$ multiplications, while $A(BC)$ needs $npq + mnq$.

a) First we try when m, n, p, q are $2, 4, 7, 10$ respectively. Here we get $2\cdot4\cdot7 + 2\cdot7\cdot10 = 196$ vs. $2\cdot4\cdot10 + 4\cdot7\cdot10 = 360$. Multiplying $(AB)C$ will be faster.

b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are N by 1, then $\mathbf{u}^{\mathrm{T}}(\mathbf{v}\mathbf{w}^{\mathrm{T}})$ takes N^2 to compute $\mathbf{v}\mathbf{w}^{\mathrm{T}}$ and another N^2 to multiply by $\mathbf{u}^{\mathrm{T}}$. On the other hand, $(\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{w}^{\mathrm{T}}$ requires N and then N; this will be much faster.

c) We want to know if $mnp + mpq < npq + mnq$. Divide everything by $mnpq$, and this is equivalent to $\frac{1}{q} + \frac{1}{n} < \frac{1}{p} + \frac{1}{m}$.
A very practical instance of this problem is in computing $AB\mathbf{v}$, where A and B are square matrices and $\mathbf{v}$ is a vector: computing it as $A(B\mathbf{v})$ is much faster. Don't forget this on an exam!

Problem 6 (§2.5, 8). a) Column 1 + column 2 - column 3 is just $A\mathbf{x}$, where $\mathbf{x}$ is the vector $\mathbf{x} = (1, 1, -1)$. The given equality means that $A\mathbf{x} = \mathbf{0}$, so A is not invertible.

b) After elimination, columns 1 and 2 end in zeroes (since elimination leaves us with an upper triangular matrix). On the other hand, adding multiples of rows to each other doesn't change the fact that column 1 + column 2 - column 3, so after elimination, column 3 ends in a zero too. There is no third pivot.

Problem 7 (§2.5, 25). Let's compute $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$ by the Gauss-Jordan method. Of course there is a formula in the 3×3 case we could use instead; you can check that it gives the same answer.

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \rightarrow \text{Phase 1 complete!} \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \rightarrow \text{Phase 2 complete!}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

The 3×3 block on the right is our answer:

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

On the other hand, B is not invertible. The sum of all of the columns is 0, i.e.

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If you run elimination, you will fail to give a third pivot.

Problem 8 (§2.5, 40). We compute the inverse of

$$A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Elimination is going to be pretty painless; the matrix is already upper-triangular, and we just need to get 0's over the pivots.

$$\to \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

So the inverse is

$$\begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 9. *The* 3×3 *matrix* A *is given as the sum of two other* 3×3 *matrices* B *and* C *satisfying:*

- *all rows of* B *are the same vector* $\mathbf{u}$.
- *all columns of* C *are the same vector* $\mathbf{v}$.

Show that A *is not invertible. One possible approach is to exlpain why there is a nonzero vector* $\mathbf{x}$ *satisfying both* $B\mathbf{x} = 0$ *and* $C\mathbf{x} = 0$*, so that* $A\mathbf{x} = (B + C)\mathbf{x} = B\mathbf{x} + C\mathbf{x} = 0$ *has a nonzero solution.*

Let's try to follow the hint. What vector might this $\mathbf{x}$ be?

To get $B\mathbf{x} = 0$, we need to have $\mathbf{r} \cdot \mathbf{x} = 0$, where $\mathbf{r}$ is any row of $\mathbf{x}$. All the rows of B are the same, so this will be the case as long as $\mathbf{x}$ is perpendicular to the single vector $\mathbf{u}$. There are lots of such vectors $\mathbf{x}$ – anything lying in the plane normal to $\mathbf{u}$ will do the trick.

For $C\mathbf{x} = 0$, we need $(v_i, v_i, v_i) \cdot \mathbf{x} = 0$, where v_i is the i^{th} entry of $\mathbf{v}$. This will be the case as long as $(1, 1, 1) \cdot \mathbf{x} = 0$.

Putting this together, we're in good shape if the vector $\mathbf{x}$ is perpendicular to the two vectors $\mathbf{u}$ and $(1, 1, 1)$. The vectors perpendicular to $\mathbf{u}$ constitute of a plane, as do those perpendicular to $(1, 1, 1)$. These planes must intersect in a line (both go through $(0, 0, 0)$, so they aren't parallel), and any nonzero $\mathbf{x}$ in the intersection will work. Alternately, we actually know how to find such a vector – just use $\mathbf{x} = \mathbf{u} \times (1, 1, 1)$! The cross product of two nonzero vectors is nonzero, so we're now done using the argument in the hint.

Problem 10. *Construct a tridiagonal matrix A with pivots 3, 4, and 5 so that performing elimination steps on A goes:*

- *subtract row* 1 *from row* 2*.*
- *subtract row* 2 *from row* 3*.*

Let's reason it out. Say our matrix is

$$\begin{bmatrix} 3 & a & 0 \\ a & b & c \\ 0 & c & d \end{bmatrix}$$

(The problem doesn't actually ask for our matrix to be symmetric, but we'll do it anyway.) For the first step to be "subtract row 1 from row 2", we'd better have $a = 3$. But then $b - 3$ is the next pivot, which should be 4. So $b = 7$ and our matrix is $\left(\begin{smallmatrix} 3 & 3 & 0 \\ 3 & 7 & c \\ 0 & c & d \end{smallmatrix}\right)$. After the first step we have

$$\begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & c \\ 0 & c & d \end{bmatrix}$$

Now it must be that $c = 4$, and as before $d = 9$. So use the matrix

$$M = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 7 & 4 \\ 0 & 4 & 9 \end{bmatrix}.$$

Elimination goes

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 7 & 4 \\ 0 & 4 & 9 \end{bmatrix} \to \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 9 \end{bmatrix} \to \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$