18.06 Final Exam Professor Strang May 22, 2013

Please PRINT your name ______________________________

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r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
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r6	T	1	36-153	Stephen Curran					

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(1) **(7+7 pts)**

(a) Suppose the nullspace of a square matrix A is spanned by the vector $v = (4, 2, 2, 0)$. Find the reduced echelon form $R = \text{rref}\,(A)$.

(b) Suppose S and T are subspaces of $\mathbf{R}^5$ and Y and Z are subspaces of $\mathbf{R}^3$. When can they be the four fundamental subspaces of a 3 by 5 matrix B? Find any required conditions to have $S = C(B^T),\ T = N(B),\ Y = C(B),$ and $Z = N(B^T)$.

(2) **(6+6 pts.)**

(a) Find bases for all four fundamental subspaces of this R.

$$R = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find U, Σ, V in the Singular Value Decomposition $A = U\Sigma V^T$:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

(3) **(5+5 pts.)**

Suppose $q_1, \ldots q_5$ are orthonormal vectors in $\mathbf{R}^5$.

The 5 by 3 matrix A has columns q_1, q_2, q_3.

(a) If $b = q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5$, find the best least squares solution $\widehat{x}$ to $Ax = b$.

(b) If terms of q_1, q_2, q_3 find the projection matrix P onto the column space of A.

(4) **(3+4+3 pts.)**

The matrix A is symmetric and also orthogonal.

(a) How is A^{-1} related to A?

(b) What number(s) can be eigenvalues of A and why?

(c) Here is an example of A. What are the eigenvalues of this matrix? I don't recommend computing with $\det(A - \lambda I)$! Find a way to use part (b).

$$A = \begin{bmatrix} .5 & -.5 & -.1 & -.7 \\ -.5 & .5 & -.1 & -.7 \\ -.1 & -.1 & .98 & -.14 \\ -.7 & -.7 & -.14 & .02 \end{bmatrix}.$$

(5) **(4+5+3 pts.)**

Suppose the real column vectors q_1 and q_2 and q_3 are orthonormal.

(a) Show that the matrix $A = q_1 q_1^T + 2q_2 q_2^T + 5q_3 q_3^T$ has the eigenvalues $\lambda = 1,\ 2,\ 5$.

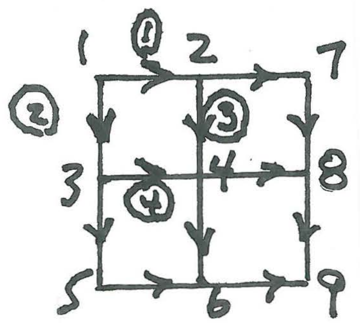
(b) Solve the differential equation $du/dt = Au$ starting at any vector $u(0)$. Your answer can involve the matrix Q with columns $q_{1,}\ q_{2,}\ q_{3,}$.

(c) Solve the differential equation $du/dt = Au$ starting from $u(0) = q_1 - q_3$.

(6) **(4+3+3 pts.)**

This graph has $m = 12$ edges and $n = 9$ nodes. Its 12 by 9 incidence matrix A has a single -1 and $+1$ in every row, to show the start and end nodes of the corresponding edge in the graph.

(a) Write down the 4 by 4 submatrix S of A that comes from the 4-node graph (a loop) in the corner. Find a vector x in the nullspace $N(S)$ and a vector y in $N(S^T)$.



(b) For the whole matrix A, find a vector Y in $N(A^T)$. You won't need to write A or to know more edge numbers.

(c) The all-ones vector $(1, 1, \ldots, 1)$ spans $N(A)$. Find the dimension of the left nullspace $N(A^T)$ (give a number).

(7) **(3+3+3+3 pts.)**

The equation $y_{n+2} + By_{n+1} + Cy_n = 0$ has the solution $y_n = \lambda^n$ if $\lambda^2 + B\lambda + C = 0$. In most cases this will give two roots λ_1, λ_2 and the complete solution is $y_n = c_1\,\lambda_1^n + c_2\,\lambda_2^n$.

Now solve the same problem the matrix way (slower). Create this vector unknown and vector equation $u_{n+1} = A\,u_n$.

$$u_n = \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix}.$$

(a) What is the matrix A in that equation?

(b) What equation gives the eigenvalues λ_1 and λ_2 of A?

(c) If λ_1 is an eigenvalue, show directly that

$$A \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{so we have the eigenvector.}$$

(d) If $\lambda_1 \neq \lambda_2$, what is now the complete solution u_n (including constants c_1 and c_2) to our equation $u_{n+1} = A\,u_n$? ((Then y_n is the first component of u_n.))

(8) **(5+5 pts.)**

(a) Find the determinant of this matrix A, using the cofactors of row 1.

$$A = \begin{bmatrix} 1 & b & 0 & 0 \\ b & 1 & b & 0 \\ 0 & b & 1 & b \\ 0 & 0 & b & 1 \end{bmatrix}.$$

(b) Find the determinant of A by the BIG formula with 24 terms. This means to find all the nonzero terms in that formula with their correct signs.

(9) **(6+4 pts.)**

(a) Suppose $\mathbf{v} = (v_1, v_2, v_3)$ is a column vector, so $A = \mathbf{v}\mathbf{v}^T$ is a symmetric matrix. Show that A is positive semidefinite, using one of these tests:

1. The eigenvalue test
2. The determinant test
3. The energy test on $x^T Ax$.

(b) Suppose A is m by n of rank r. What conditions on m and n and r guarantee that $A^T A$ is positive definite? If those conditions fail, prove that $A^T A$ **will not be positive definite.**