SOLUTIONS TO EXAM 2

Problem 1 (30 pts)

(a) The rank of P is 2. Any vector perpendicular to the subspace spanned by a_1 and a_2 is in the nullspace of P, and the orthogonal complement of the subspace spanned by a_1 and a_2 is 3-dimensional (that is, there are three independent vectors that project to 0 by P). This is exactly the nullspace of P, and since

$$\text{rank } P = \dim C(P) = 5 - \dim \text{Nullspace } P,$$

the rank of P is $5 - 3 = 2$.

(b) The nullspace of P is the left nullspace of A. Indeed, we have

$$\begin{aligned} Pv = 0 &\Leftrightarrow a_1^T v = 0 \text{ and } a_2^T v = 0 \\ &\Leftrightarrow v^T a_1 = 0 \text{ and } v^T a_2 = 0 \\ &\Leftrightarrow vA = 0. \end{aligned}$$

(c) Gram-Schmidt gives

$$q_1 = \frac{a_1}{||a_1||} = \frac{(1,0,1,0,4)^T}{\sqrt{1^2+0^2+1^2+0^2+4^2}} = \frac{1}{3\sqrt{2}}(1,0,1,0,4)^T$$

and

$$\begin{aligned} q_2 &= \frac{a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1}{||a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1||} = \frac{a_2 - a_2^T q_1 q_1}{||a_2 - a_2^T q_1 q_1||} = \frac{(2,0,0,0,4)^T - (1,0,1,0,4)^T}{||(2,0,0,0,4)^T - (1,0,1,0,4)^T||} \\ &= \frac{1}{\sqrt{2}}(1,0,-1,0,0)^T, \end{aligned}$$

and q_1 and q_2 form an orthonormal basis for the column space of A.

(d) Since P is a projection matrix, we have $P = P^T$. To show that Q is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$\begin{aligned} QQ^T &= (I-2P)(I-2P)^T \\ &= (I-2P)(I^T - 2P^T) \\ &= (I-2P)(I-2P)(\ I \text{ and } P \text{ are symmetric}) \\ &= I - 4P + 4P^2 \end{aligned}$$

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

Problem 2 (30 pts)

(a) We will find the determinant by doing row operations:

$$\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and the last matrix has determinant $(1) \cdot (2) \cdot (3) \cdot (4) = 24$, so the original matrix has determinant -24.

(b) $\det A$ tells the volume of a box in $\mathbb{R}^4$ whose sides are given by the vectors $(1,1,0,0)^T$, $(2,2,2,0)^T$, $(0,3,3,3)^T$, and $(0,0,4,4)^T$. Another box with the same volume would be a box whose sides are given by the vectors $(1,0,0,0)^T$, $(2,2,0,0)^T$, $(0,3,3,0)^T$, and $(0,4,0,4)^T$. (these are obtained from A via row operations, and so the absolute value of the determinants do not change!)

(c) The formula for A^{-1} says that (see page 270 of the textbook!)

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

where C_{ji} is the cofactor given by removing row j and column i. From the problem, this matrix is not invertible, so its determinant is 0, meaning that $C_{ij} = 0$. This means that the $(4,3)$-entry of A^{-1} is also 0.

Problem 3 (30 pts)

(a) Letting

$$A = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix},$$

the projection matrix that projects every $b \in R^4$ onto the column space of A (which is the line through q_4) is given by the formula

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(b) Letting

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

the projection matrix that projects every $b \in R^4$ onto the column space of A (which is the subspace spanned by q_1, q_2 and q_3) is given by the formula

$$A(A^TA)^{-1}A^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

(c) We must solve the new system

$$A^TA\hat{x} = A^Tb.$$

Since $A^TA = I$, we have

$$\hat{x} = A^Tb = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}.$$

Then $A\hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $e = b - A\hat{x} = 0$.