Problem Set 3 Solutions:

Section 6.4

6.

We need to the columns of Q to be an orthonormal basis of eigenvectors of A. This gives eight choices:

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}$$

10. If x is not real then even if A is real there is not reason to expect that $x^T x$ or $x^T Ax$ is real, so this "proof" makes no sense.

14. This matrix M is skew-symmetric and also orthogonal. Thus $M^T M = -M^2 = I$ so the eigenvalues of M can only be i and $-i$. The sum of the eigenvalues of M is the trace of M so the eigenvalues of M must be $-i$, $-i$, i, and i.

15. The characteristic polynomial $det(A - \lambda I)$ of A is $(\lambda - i)(\lambda + i) - 1 = \lambda^2$ so 0 is the only eigenvalue of A and it has algebraic multiplicity 2.

Solving $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ gives that $x_1 - ix_2 = 0$, so the solutions have the form $c \begin{bmatrix} i \\ 1 \end{bmatrix}$

The eigenvalue 0 of A has geometric multiplicity 1 so A is not diagonalizable.

23. A is an invertible, orthogonal, permutation, diagonalizable, and Maarkov matrix. A does not have an LU factorization but it does have a QR, $S\Lambda S^{-1}$, and $Q\Lambda Q^{-1}$ factorization.

B is a projection, diagonalizable, and Markov matrix. It has an LU, QR, $S\Lambda S^{-1}$, and $Q\Lambda Q^{-1}$ factorization.

Note: The book says that B does not have an LU or QR factorization, possibly because it wants U and R to be invertible for these factorizations, although I do not see why this is required. We can write

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

24. $A = Q\Lambda Q^{-1}$ is possible when $b = 1$ and A is symmetric. $A = S\Lambda S^{-1}$ is impossible when $b = -1$. This means the characteristic polynomial of A is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ but $A \neq I$. A is not invertible when $b = 0$.

Section 6.5

2.

A_1 has negative determinant, so it fails the test. Taking $x = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$, $x^T A_1 x < 0$.

$(A_2)_{11} < 0$, so A_2 fails the test.

$(A_3)_{11} > 0$ and $|A_3| = 0$, so A_3 is positive semidefinite but not positive definite.

$(A_4)_{11} > 0$ and $|A_4| = 1$, so A_4 is positive definite and has two positive eigenvalues.

16. $x^T Ax < 0$ when $(x_1, x_2, x_3) = (1, -5, 0)$

Note: $(0, 1, 0)$ will probably be the most common correct answer.

21. The conditions on the upper left determininants of A are:

1. $s > 0$
2. $s^2 - 16 > 0$
3. $s(s^2 - 16) + 4(-4s - 16) - 4(16 + 4s) = s^3 - 48s - 128 > 0$

These three conditions give $s > 0$, $s > 4$, and $s > 8$ respectively. Thus we need $s > 8$

The conditions on the upper left determinants of B are:
1. $t > 0$
2. $t^2 - 9 > 0$ 3. $t(t^2 - 16) - 3(3t) = t^3 - 25t > 0$
These three conditions give $t > 0$, $t > 3$, and $t > 5$ respectively. Thus we need $t > 5$
Note: The problem can also be answered by saying that s, t must be bigger than -1 times the smallest eigenvalue of the matrices A and B respectively.

28.
a. 10
b. 2,5
c. $\begin{bmatrix} cos(\Theta) \\ sin(\Theta) \end{bmatrix}, \begin{bmatrix} -sin(\Theta) \\ cos(\Theta) \end{bmatrix}$
d. A is of the form $Q\Lambda Q^{-1}$ where Q is orthogonal and Λ is diagonal, so A is symmetric. A has only positive eigenvalues, so it is positive definite.

35. If $x \neq 0$ then $Ax \neq 0$ so $x^T A^T CAx = (Ax)^T C(Ax) > 0$ as C is positive definite. Thus $A^T CA$ is positive definite, as claimed.

Section 6.3, Problem 30. (non-MATLAB solution)
a.
The inverse of the left matrix is
$\frac{1}{1+(\frac{\Delta t}{2})^2} \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix}$
We have $A = \frac{1}{1+(\frac{\Delta t}{2})^2} \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-(\frac{\Delta t}{2})^2}{1+(\frac{\Delta t}{2})^2} & \frac{\Delta t}{1+(\frac{\Delta t}{2})^2} \\ -\frac{\Delta t}{1+(\frac{\Delta t}{2})^2} & \frac{1-(\frac{\Delta t}{2})^2}{1+(\frac{\Delta t}{2})^2} \end{bmatrix}$
The columns of A are clearly orthogonal. To see that they have unit norm, note that
$(\frac{1-(\frac{\Delta t}{2})^2}{1+(\frac{\Delta t}{2})^2})^2 + (\frac{\Delta t}{1+(\frac{\Delta t}{2})^2})^2 = \frac{1-\frac{(\Delta t)^2}{2}+(\frac{\Delta t}{2})^4+(\Delta t)^2}{1+\frac{(\Delta t)^2}{2}+(\frac{\Delta t}{2})^4} = 1$
If $B^T = -B$ then if $A = (I - B)^{-1}(I + B)$, $A^T = (I + B)^T((I - B)^T)^{-1} = (I - B)(I + B)^{-1}$
Then $AA^T = (I - B)^{-1}(I + B)(I - B)(I + B)^{-1} = (I - B)^{-1}(I - B^2)(I + B)^{-1} =$
$(I - B)^{-1}(I - B)(I + B)(I + B)^{-1} = I$
b.
A is the matrix corresponding to clockwise rotation by $\Theta = \sin^{-1}(\frac{\Delta t}{1+(\frac{\Delta t}{2})^2}) \approx .1957$
Rotating $(1, 0)$ clockwise by 32Θ gives approximately $(.9998, .0201)$

Section 8.1, problem 11. (setup)
This differential equation has the exact solution
$u(x) = c_1 + \frac{x}{10} + c_2 e^{10x}$
Solving $u(0) = u(1) = 0$ gives $c_1 + c_2 = 0$, $c_1 + \frac{1}{10} + e^{10}c_2 = 0$
$\begin{bmatrix} 1 & 1 \\ 1 & e^{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{10} \end{bmatrix}$
$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{e^{10}-1} \begin{bmatrix} e^{10} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{10(e^{10}-1)} \\ -\frac{1}{10(e^{10}-1)} \end{bmatrix}$
$u(x) = \frac{1}{10(e^{10}-1)} + \frac{x}{10} - \frac{e^{10x}}{10(e^{10}-1)}$
For the numerical approximation, let $u_n = u(n\Delta x)$. Then our condition says that $u_0 = u_8 = 0$.

Let $A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

Let $B_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

Let $B_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$

Let $B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}$

The equations we need to solve are

$$-64Au + 80B_iu = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where $i = 1$ for the forward differences, 2 for the centered differences, and 3 for the backwards differences.