Problem Set 3 Solutions:

Section 3.2

6.

Elimination on A gives the row reduced echelon matrix $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

The free variables are x_2 and x_3 so the special solutions to $Rx = 0$ are $(3,1,0)$ and $(5,0,1)$.

Elimination on B gives the row reduced echelon matrix $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The free variable is x_2 so the special solution to $Rx = 0$ is $(3,1,0)$.

For an m by n matrix the number of pivot variables plus the number of free variables is n.

14. If the first and last columns of a 3 by 5 matrix are the same and nonzero, then x_5 is a free variable. The special solution corresponding to x_5 is $(-1,0,0,0,1)$.

15. Suppose an m by n matrix has r pivots. The number of special solutions is $n - r$. The nullspace contains only $x = 0$ when $r = n$. The column space is all of $\mathbb{R}^m$ when $r = m$.

18. The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12,0,0)$. All points on the plane have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Section 3.3

Problem 4:

If the matrix is an m by n matrix and all of the pivot variables come last instead of first, then the row reduced echelon form is

$$R = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

where the upper right block is r by r.

The nullspace matrix N containing the special solutions is

$$N = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

where the top block is $n - r$ by $n - r$ and the bottom block is $n - r$ by r.

28. The "row-and-column reduced form" for an m by n matrix of rank r is

$$R = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where the upper left block is r by r.

Section 3.4

3.

After elimination, we obtain the following equation

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This has complete solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

5.
After elimination, we obtain the following equation

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ b_3 - 2b_1 - b_2 \end{bmatrix}$$

This is only solvable when $b_3 = 2b_1 + b_2$. If $b_3 = 2b_1 + b_2$, this has complete solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

15. Suppose row 3 of U has no pivot. Then that row is 0. The equation $Ux = c$ cannot be solved unless $c_3 = 0$ (although it may not be solvable even if $c_3 = 0$). The equation $Ax = b$ might not be solvable.

17.
The largest possible rank of a 6 by 4 matrix is 4. Then there is a pivot in every column of U and R. The solution to $Ax = B$ is always unique if it exists. The nullspace of A is 0. An example is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Section 3.5
1.

If $c_1v_1 + c_2v_2 + c_3v_3 = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = 0$

then $c_3 = 0$ which means that c_2 must also be 0 which means that c_1 must be zero as well. Thus $c_1v_1 + c_2v_2 + c_3v_3 = 0$ if and only if $c_1 = c_2 = c_3 = 0$.
$-v_1 - v_2 + 4v_3 - v_4 = 0$ so v_1, v_2, v_3, v_4 are linearly dependent.

9. Suppose v_1, v_2, v_3, v_4 are 4 vectors in $\mathbb{R}^3$. These vectors are linearly dependent because any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$. The two vectors v_1 and v_2 will be dependent if one is a multiple of the other (note that $0 = 0v$ for any v). The vectors v_1 and $(0, 0, 0)$ are dependent because $0v_1 + (0, 0, 0) = (0, 0, 0)$

16.
a.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

d. $C(I)$ has basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$N(I)$ has empty basis (it is the zero vector space).

Problem 9:

For any subspace V of $\mathbb{R}^n$, we can do the following. Let d_V be the dimension of V and choose a basis $v_1, \cdots, v_{V_d}$ for V. Now let A be the d_V by n matrix with rows $v_1, \cdots, v_{V_d}$. Using elimination, we can obtain a d_V by n row reduced echelon matrix R whose rows form another basis for V.

R has d_V pivots so the equation $Rx = 0$ has $n - d_V$ free variables. This implies that there are $n - d_V$ special solutions to $Rx = 0$ and these solutions form a basis for $N(R) = N(A)$. Construct an n by $n - d_V$ matrix N_1 whose columns are these special solutions. N_1 is the nullspace matrix for A.

Now note that $AN_1 = RN_1 = 0$, so ${N_1}^T A^T = {N_1}^T R^T = 0$ which means that $V \subseteq N({N_1}^T)$. Because of the way N_1 was constructed, it has $n - d_V$ pivots so the equation ${N_1}^T y = 0$ has d_V free variables. This implies that $N({N_1}^T)$ has dimension d_V. $V \subseteq N({N_1}^T)$ and V has dimension d_V, so we must have that $V = N({N_1}^T)$.

V is the nullspace of an $n - d_V$ by n matrix ${N_1}^T$. Similarly, if W has dimension d_W then W is the nullspace of an $n - d_W$ by n matrix ${N_2}^T$. Letting N be the matrix obtained by putting ${N_1}^T$ on top of ${N_2}^T$, $V \cap W = N({N_1}^T) \cap N({N_1}^T) = N(N)$. N is a $2n - d_V - d_W$ by n matrix so if $d_V + d_W > n$ then N has more columns then rows which implies that $Nx = 0$ has a nonzero solution and thus $V \cap W$ contains a nonzero vector, as needed.

Problem 10.

a.

When choosing an A which has zero row sums, we can choose $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}$ and the other elements of A will be uniquely determined by our choices. Thus, one basis for the subspace of matrices with zero row sums is:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

For any matrix A,

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_{11} + A_{12} + A_{13} \\ A_{21} + A_{22} + A_{23} \\ A_{31} + A_{32} + A_{33} \end{bmatrix}$$

which is 0 for any $A \in V$. Thus $(1, 1, 1)$ is in $N(A)$ for any $A \in V$.

When choosing an A which has zero column sums, we can choose $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ and the other elements of A will be uniquely determined by our choices. Thus, one basis for the subspace of matrices with zero column sums is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

b. $V \cap W$ has dimension 4. We can choose the elements $A_{11}.A_{12}, A_{21}, A_{22}$ freely and the other elements of A will be uniquely determined, so one basis for $V \cap W$ is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

c. For all matrices $A \in V$, $A_{11} + A_{12} + A_{13} + A_{21} + A_{22} + A_{23} + A_{31} + A_{32} + A_{33} = 0$. The same relation holds for all matrices $B \in W$. Thus, for any matrix M of the form $M = A + B$ where $A \in V, B \in W$, $M_{11} + M_{12} + M_{13} + M_{21} + M_{22} + M_{23} + M_{31} + M_{32} + M_{33} = 0$. Any matrix for which this is not true cannot be written in this form. One example is the identity matrix I.