18.06 Quiz 1 Professor Strang March 9, 2012

**Please PRINT your name** ______________________________ **1.**

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1. **(36 pts.)** Suppose the 4 by 4 matrix $A$ (with 2 by 2 blocks) is already reduced to its rref form

$$A = \begin{bmatrix} I & 3I \\ 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$.

(b) Describe all possible bases for $C(A)$.

(c) Find a basis (special solutions are good) for the nullspace $N(A)$.

(d) Find the complete solution $x$ to the 4 by 4 system

$$Ax = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

*Solution.*

(a) The column space is spanned by the vectors $(1,0,0,0),(0,1,0,0),(3,0,0,0),(0,3,0,0)$. We then put them in a matrix and do a Gaussian elimination to find independent vectors. This tells us that the basis for the column space is $\{(1,0,0,0),(0,1,0,0)\}$

(b) The column space can be described by

$$C(A)=\{(x,y,0,0)\mid x,y\in\mathbb{R}\},$$

so the basis of $C(A)$ is the set of any two independent vectors $(x_1,x_2,0,0)$ and $(x_3,x_4,0,0)$. This means that the matrix

$$A=\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

has full rank (in other words $x_1x_4-x_2x_3\neq 0$ must hold).

(c) We observe that $(3,0,-1,0)$ and $(0,3,0,-1)$ are two independent vectors belonging to the null space. Since the column space has dimension 2, the null space has dimension $4-2=2$, so any basis of $N(A)$ has two elements. Hence, $\{(3,0,-1,0),(0,3,0,-1)\}$ is a basis for $N(A)$.

(d) We start by looking for $x_{\text{particular}}$ via elimination. Note that the matrix is already in a reduced row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_{\text{particular}}=(5,4,0,0)$. Then the complete solution is given by

$$\begin{aligned} x &= x_{\text{particular}}+x_{\text{nullspace}} \\ &= (5,4,0,0)+(3a,3b,-a,-b) \\ &= (5+3a,4+3b,-a,-b) \end{aligned}$$

for any $a,b\in\mathbb{R}$.

□

2. **(16 pts.)** Suppose the matrix $A$ is $m$ by $n$ of rank $r$, and the matrix $B$ is $M$ by $N$ of rank $R$. Suppose the column space $C(A)$ is contained in (possibly equal to) the column space $C(B)$. (This means that every vector in $C(A)$ is also in $C(B)$.) What relations must hold between $m$ and $M$, $n$ and $N$, and $r$ and $R$?

It might be good to write down an example of $A$ and $B$ where all the columns are different.

*Solution.* The column space of $A$ is contained in $\mathbb{R}^m$, and the column space of $B$ is contained in $\mathbb{R}^M$. If $C(A) \subseteq C(B)$, this means they are contained in the same Euclidean space, so $M = m$. The dimension of the column space is the rank of the matrix, so if $C(A) \subseteq C(B)$, this means $\dim C(A) \leq \dim C(B)$, hence $r \leq R$. There are no relations between $N$ and $n$; $n = N$ if $A = B$, $n \leq N$ if $B = [A\ A]$, and $n \geq N$ if $A = [B\ B]$.

□

3. (a) **(16 pts.)** Suppose three matrices satisfy $AB = C$. If the columns of $B$ are dependent, show that the columns of $C$ are dependent.

(b) **(12 pts.)** If $A$ is 5 by 3 and $B$ is 3 by 5, show using part (a) or otherwise that $AB = I$ is impossible.

*Solution.* (a) The columns of $B$ being dependent means by definition that there is a vector $\mathbf{x} \neq 0$ such that $B\mathbf{x} = 0$. But then we also have

$$C\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0},$$

which means that the same $\mathbf{x} \neq 0$ works to show that the columns of $C$ are dependent.

(b) The columns of $B$ are dependent, since these are five vectors in $\mathbb{R}^3$, and $5 > 3$. Thus, by part (a), the columns of $AB$ must be dependent. However, columns of $I$ are independent, so $AB$ can never equal $I$. [Note: Switching the order matters here. One can indeed find a $3 \times 5$ matrix A, and a $5 \times 3$ matrix $B$ such that $AB = I$ is the $3 \times 3$ identity - hence any "proof" that is insensitive to the order of $A$ and $B$ must be flawed].

□

4. **(20 pts.)** Apply row elimination to reduce this invertible matrix from $A$ to $I$. Then write $A^{-1}$ as a product of three (or more) simple matrices coming from that elimination. Multiply these matrices to find $A^{-1}$.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

*Solution.* Swapping rows 1 and 2 corresponds to

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Subtracting 4 times row 1 from row 3 corresponds to

$$E_{31} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

Subtracting row 3 from row 2 corresponds to

$$E_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Putting them together, we get

$$E_{23}E_{31}PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = I.$$

Hence, $A^{-1} = E_{23}E_{31}P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -4 & 1 \end{pmatrix}.$

□