18.06 Spring 2012 – Problem Set 9

This problem set is due Thursday, May 3rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 5 & 11 from Section 6.6.

   Solution. **Problem 5.** $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&1\end{bmatrix}$, $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\0&1\end{bmatrix}$ are similar (they all have eigenvalues 1 and 0).

   $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is by itself (eigenvalue 1), and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ is also by itself, with eigenvalues 1 and -1.

   Problem 11. $\mathbf{u}(0) = \begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}v(0)\\w(0)\end{bmatrix}$. The equation

   $$\frac{d\mathbf{u}}{dt} = \begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}\mathbf{u}$$

   has

   $$\frac{dv}{dt} = \lambda v + w \qquad \text{and} \qquad \frac{dw}{dt} = \lambda w.$$

   Then $w(t) = 2e^{\lambda t}$ and $v(t)$ must include $2te^{\lambda t}$ (this comes from the repeated λ). To match $v(0) = 5$, the solution is $v(t) = 2te^{\lambda t} + 5e^{\lambda t}$. □

2. Do Problems 17 & 19 from Section 6.6.

   Solution. **Problem 17**

   (a) *False:* Diagonalize a nonsymmetric matrix $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar.

   (b) *True:* A singular matrix has $\lambda = 0$.

   (c) *False:* $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ are similar (they have $\lambda = \pm 1$).

   (d) *True:* Adding I increases all eigenvalues by 1.

   Problem 19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeroes. □

3. Do Problem 22 from Section 6.6.

   Solution. Let $J = M^{-1}AM$ be the Jordan form of A. Since J is a strictly upper triangular matrix, $J^n = 0$. Hence $A^n = (M^{-1}JM)^n = M^{-1}J^nM = 0$. □

4. Do Problems 3 & 6 from Section 6.7.

Solution. **Problem 3** There is a orthogonal matrix V such that $VA^TAV^T = (VA^T) \cdot (VA^T)^T = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Let $a_1 = [a_{11}a_{12}]$ and $a_2 = [a_{21}a_{22}]$. Since orthogonal matrices preserve length of vectors,

$$\sigma^2 + 0^2 = ||Va_1^T||^2 = ||a_1^T||^2 = a_{11}^2 + a_{12}^2$$

and

$$0^2 + 0^2 = ||Va_2^T||^2 = ||a_2^T||^2 = a_{21}^2 + a_{22}^2.$$

Hence $\sum a_{ij}^2 = \sigma^2$.

Problem 6

The eigenvalues of $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are 3 and 1. The corresponding normal eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Hence we have $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

The eigenvalues of $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ are 3, 1 and 0. The corresponding normal eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, and $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$. Hence we have $V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Hence

$$A = U \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} V^T.$$

□

5. Do Problems 9 & 10 from Section 6.7.

Solution. **Problem 9** Since A is a orthogonal matrix, $AA^T = A^TA = I$. The only eigenvalue of I is 1, so $A = UIV^T = UV^T$.

Problem 10 Note that $A = U\Sigma V^T$ where U and V are orthogonal matrices with first columns u and v, respectively, and $\Sigma = \begin{bmatrix} 12 & \\ & \end{bmatrix}$. Our matrix $A = 12uv^T$ and the only singular value if $\sigma_1 = 12$.

□

6. Do Problem 14 from Section 6.7.

Solution. Let $A = U\Sigma V^T = Udiag(\sigma_1, \cdots, \sigma_n)V^T$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. The closet singular matrix is $A_0 = U\Sigma_0 V^T = Udiag(\sigma_1, \cdots, \sigma_{n-1}, 0)V^T$. □

7. Do Problems 1 & 4 from Section 8.5.

Solution. **Problem 1** If $j \neq k$,the integral of RHS is

$$\int_0^{2\pi} \cos(j+k)x + \cos(j-k)x dx = \frac{\sin(j+k)x}{j+k} + \frac{\sin(j-k)x}{j-k}\Bigg|_{x=0}^{2\pi} = 0.$$

Hence $(\cos jx, \cos kx) = 0$. If $j = k$,

$$\int_0^{2\pi} \cos(j+k)x + \cos(j-k)x dx = \frac{\sin(j+k)x}{j+k} + 1\Bigg|_{x=0}^{2\pi} = 2\pi.$$

So $(\cos jx, \cos kx) = \pi$.

Problem 4 For any c, $x^3 - cx$ is orthogonal to 1 and $x^2 - \frac{1}{3}$, and

$$\int_{-1} 1x \cdot (x^3 - cx)dx = \frac{2}{5} - 2\frac{c}{3}.$$

If we let $c = \frac{3}{5}$, then $x^3 - cx$ is orthogonal to every other functions.

□

8. Do Problems 3 & 19 from Section 10.2.

Solution. **Problem 3** $z = C[-1-i, -1-i, 2]^T$ for some constant C. Since $z^H A^H = 0$, $z\overline{a_i} = 0$ for any column a_i of A. However, $z \cdot [-i, 1, -i] = -1 - i$. Hence columns of A^T are not orthogonal to z.

Problem 19 Note that $e^{-ix} = \cos x - i \sin x$ and $e^{ix} = \cos x + i \sin x$. Their inner product is

$$\int_0^{2\pi} e^{-ix}\overline{e^{ix}}dx = \int_0^{2\pi} \cos^2 x - \sin^2 x - 2i \cos x \sin x dx = \int_0^{2\pi} \cos 2x - i \sin 2x dx = 0.$$

□

9. Do Problem 7 from Section 10.3.

Solution.

$$c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = Fc.$$

Also,

$$C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = FC.$$

□

10. Do Problem 15 from Section 10.3.

    Solution. Diagonal E needs n multiplications, while the Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times x. $\square$

18.06 Wisdom. To reiterate: You should get as ready as you can for Exam 3, and for the finals, by doing as many old exams as you have time for (found on the 18.06 website under "Past Courses"). Get into a good rhythm with these rehearsals to stay fully tuned in on Linear Algebra, and to master the concepts - you will need it, both here and beyond.