18.06 Spring 2012 – Problem Set 8

This problem set is due Thursday, April 26th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 3 from Section 6.4.

   Solution. Solve $det(A-\lambda I) = 0$ to get the eigenvalues $0, -2, 4$. Then find unit vectors in the nullspaces of the matrices $A - 0I, A + 2I, A - 4I$ to get the unit eigenvectors $\frac{1}{\sqrt{2}}[0, 1, -1]^T$, $\frac{1}{\sqrt{3}}[1, -1, -1]^T$, and $\frac{1}{\sqrt{6}}[2, 1, 1]^T$. □

2. Do Problem 22 from Section 6.4 (now a paradox for you).

   Solution. The flaw is that the correspondence between eigenvalues and eigenvectors need not be the same for A and A^T. For example, the matrix

   $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   has eigenvalues i and $-i$ with corresponding eigenvectors $[1, i]^T$ and $[i, 1]^T$, while the transpose

   $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

   has eigenvalues i and $-i$ with corresponding eigenvectors $[i, 1]^T$ and $[1, i]^T$. □

3. Do Problem 23 from Section 6.4.

   Solution. A is invertible, orthogonal, permutation, diagonalizable, Markov. A has possible factorizations QR, $S\Lambda S^{-1}$ and $Q\Lambda Q^T$.

   B is projection, diagonalizable and Markov. B has possible factorizations LU, QR, $S\Lambda S^{-1}$ and $Q\Lambda Q^T$; note that the matrices U and R have only one non-zero diagonal entry. □

4. Do Problem 30 from Section 6.4.

   Solution. From the decomposition $A = Q\Lambda Q^T$ we have $a_{11} = \sum_i \lambda_i q_{1i}^2$. From here, we get $a_{11} \leq \sum_i \lambda_{max} q_{1i}^2 = \lambda_{max} \sum_i q_{1i}^2 = \lambda_{max}$ □

5. Do Problem 2 & 14 from Section 6.5.

Solution. **Problem 2** Only A_4 has two positive eigenvalues. The vector $x = [-7, 6]^T$ has $x^T Ax = -7 < 0$.

Problem 14 The eigenvalues of A^{-1} are positive because they are inverses of the eigenvalues of A. For the 2 by 2 case, one can also check directly that the entries of A^{-1} pass the determinant tests. □

6. Do Problem 28 from Section 6.5.

   Solution. Use the product formula for determinants to get:

   $$det(A) = 1 \times 10 \times 1 = 10.$$

   We recognize that A is factorised in the form $Q\Lambda Q^T$. Thus it has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$, and corresponding eigenvectors $x_1 = (\cos\theta, \sin\theta)^T$ and $x_2 = (-\sin\theta, \cos\theta)$. Both eigenvalues are positive, so the matrix is positive definite. □

7. Do Problem 31 from Section 6.5.

   Solution. Complete the square to get

   $$z = 4x^2 + 12xy + cy^2 = (2x + 3y)^2 + (c - 9)y^2.$$

   Thus for $c > 9$ the graph of z is a bowl, and for $c < 9$ the graph has a saddle point. When $c = 9$, the graph, $z = (2x + 3y)^2$, is a "trough" staying at zero along the line $2x + 3y = 0$. □

8. Do Problem 35 from Section 6.5.

   Solution. "Factor" the tranposes to get

   $$x^T A^T CAx = (Ax)^T C(Ax).$$

   Since C is assumed positive definite, this energy can drop to zero only when $Ax = 0$. Since A is assumed to have independent columns, $Ax = 0$ only happens when $x = 0$. Thus $A^T CA$ has positive energy and is positive definite. □

9. Do Problem 7 from Section 8.1.

   Solution. (To first of the above) For 5 springs and 4 masses, the 5 by 4 matrix A has two non-zero diagonals: all $a_{ii} = 1$ and $a_{a+1,i} = -1$. With $C = \texttt{diag}(c_1, c_2, c_3, c_4, c_5)$, we get $K = A^T CA$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix, and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $Ku = \texttt{ones}(4, 1)$. (K^{-1} will solve $Ku = \texttt{ones}(4)$.) □

10. Do Problem 11 from Section 8.1. There was a book version/numbering issue. Here are the two "problems 8.1.11" in question:

11 (MATLAB) Find the displacements $u(1), \dots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of $\boldsymbol{u}$ with **fixed-fixed** and **fixed-free** ends. Note that diag(ones(n, 1), d) is a matrix with n ones along diagonal d. This print command will graph a vector u:

```
plot(u, '+');    xlabel('mass number');    ylabel('movement');    print
```

12 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\Delta x = \frac{1}{8}$. Graph your three numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\frac{du}{dx} = 1 \quad \text{with} \quad u(0) = u(1) = 0.$$

This ***convection-diffusion equation*** appears everywhere. It transforms to the Black-Scholes equation for option prices in mathematical finance.

Problem 12 is developed into the first MATLAB homework in my 18.085 course on Computational Science and Engineering at MIT. Videos on *ocw.mit.edu*.

Solution. To "11" in the above snippet: The two graphs of 100 points are "discrete parabolas" starting at $(0, 0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

To "12" in the above snippet: Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB:

```
E = diag(ones(6,1));
K = 64 * (2 * eye(7) - E - E');
D = 80 * (E - eye(7));
K + D\ones(7,1); % forward
(K - D')\ones(7,1); % backward
(K + D/2 - D'/2)\ones(7,1); % centered is usually the best : more accurate
```

□

18.06 Wisdom.

- Your exams so far make up 30% of your final grade. If you wish to maximize your performance in 18.06, it is all about practice (without this, you won't know what you don't know either). Make a weekly study plan involving doing as many old exams as you can fit in your schedule (find them on the 18.06 web). Organize your work. Be systematic. Learn to check your results. Keep a list of all questions (and confusing concepts) you encounter on your journey, and ask your TAs to explain all the hows and whys. We are here for the same!

- Also try out the great Math Learning Center, that you can visit and ask the instructors your questions in a friendly atmosphere, and at evening hours too: *Room 2-102, Mon-Thu, 3:00–5:00pm and 7:30–9:30pm.*