18.06 Spring 2012 – Problem Set 6

This problem set is due Thursday, April 5th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 9 & 15 from Section 5.1.

   Solution. **Problem 9**

   $\det(A) = 1$: exchange row 1 and row 3, and then row 1 and row 2.

   $\det(B) = 2$: subtract rows 1 and 2 from row 3 then columns 1 and 2 from column 3.

   $\det(C) = 0$: the rows are equal. (Note that $C = A + B$, but $\det(C) \neq \det(A) + \det(B)$.)

   Problem 15

   The first determinant is zero: subtract row 2 from row 3, and row 1 from row 2, to get a matrix with two equal rows.

   The second determinant is $(1-t^2)^2 = 1 - 2t^2 + t^4$: subtract t times row 2 from row 1, and t times row 3 from row 2, to get a lower-triangular matrix. □

2. Do Problems 18 & 22 from Section 5.1.

   Solution. **Problem 18**

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

   where to reach the 2×2 determinant, we eliminate a and a^2 in row 1 by column operations. Now factor out $b-a$ and $c-a$ from the 2×2 determinant:

$$(b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$$

   Problem 22

   $\det(A) = 3$, $\det(A^{-1}) = 1/3$, $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. □

3. Do Problems 8 & 9 from Section 5.2.

Solution. **Problem 8**

Some term $a_{1i_1}a_{2i_2}\dots a_{ni_n}$ in the big formula is not zero. Move rows $1, 2, \dots, n$ into rows i_1, i_2, $\dots$, i_n. Then these non-zero a's will be on the main diagonal.

Problem 9

There are 6 terms in the big formula, all ± 1. Thus, the determinant must be an *even* integer.

To get $+1$ for all the three product terms corresponding to the even permutations, the matrix needs to have an *even* total number of -1 entries (this is easy to see in this 3×3 situation, where $3 \cdot 3!/2 = 3^2$ happens to hold, so each matrix entry shows up exactly once somewhere in the 3 product terms coming from the even permutations). On the other hand, to also get $+1$ for all the product terms corresponding to the odd permutations, the matrix would need to have an *odd* total number of -1 entries.

Thus at least one term in the big formula must be a -1, and the maximal determinant is $+4$. Namely, this is attained for example for the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

□

4. Do Problem 20 from Section 5.2.

Solution. $G_2 = -1$, $G_3 = 2$, and $G_4 = -3$. We guess that $G_n = (-1)^{n-1}(n-1)$. □

5. Do Problem 29 from Section 5.2.

Solution. There are five non-zero products, all ± 1. Here are the $(row, column)$ coordinates of the terms, and the signs:

$$\begin{array}{cl} + & (1,1)(2,2)(3,3)(4,4) \\ + & (1,2)(2,1)(3,4)(4,3) \\ - & (1,2)(2,1)(3,3)(4,4) \\ - & (1,1)(2,2)(3,4)(4,3) \\ - & (1,1)(2,3)(3,2)(4,4) \end{array}$$

Total: -1. □

6. Do Problem 34 from Section 5.2.

Solution. (a) Consider the 3 by 5 matrix formed by the last three rows of A. It has only two non-zero columns, and so it has rank at most 2. Therefore, the last three rows of A are linearly dependent.

(b) Consider a term in the big formula for $\det A$; it is a product of entries of A, one entry in each row and column. Consider the entries coming from the last three rows of A; there are three of them, and at most two can be in the last two columns of A. Therefore, at least one entry falls in the 3 by 3 block of zeroes, and so the whole term of the big formula is 0.

□

7. Do Problems 4 & 8 from Section 5.3.

Solution. **Problem 4**

(a) We get the familiar formula $x_1 = |\mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3| / \det(A)$

(b) We use linearity of the determinant in the first column:

$$|x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_3| = x_1|\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3| + x_2|\mathbf{a}_2\ \mathbf{a}_2\ \mathbf{a}_3| + x_3|\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_3|$$

The last two terms are zero because two of the columns are the same.

Problem 8 The cofactor matrix is

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

Then $AC^T = 3I$ so $\det(A) = 3$. If you change that 4 to 100, $\det(A)$ is unchanged because the corresponding cofactor is 0.

□

8. Do Problems 20 & 25 from Section 5.3.

Solution. **Problem 20** The rows of the Hadamard matrix generate a 4-dimensional hypercube with side length 2. Therefore $\det(H) = \pm 16$. It turns out that the rows of H are in such an order that in fact $\det(H) = 16$.

Problem 25 An n dimensional cube has 2^n corners, $n\,2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. The cube in $\mathbb{R}^n$ generated by the rows of $2I$ has volume 2^n.

□

9. Do Problem 1 from Section 6.1.

Solution. The eigenvalues of A are 1 and 1/2; the eigenvalues of A^2 are 1 and 1/4; the eigenvalues of A^∞ are 1 and 0.

(a) If we swap the rows of A, the resulting matrix has eigenvalues 1 and -1/2

(b) A matrix has a zero eigenvalue if and only if it has a non-trivial nullspace. The nullspace of a matrix is not changed by the steps of elimination; therefore, a zero eigenvalue is not changed by the steps of elimination.

□

10. Use MATLAB to "prove" all the facts you remember (or may not remember?) about determinants. First define the following matrices to test on (copy paste into MATLAB - retyping is time- and patience-consuming):

```
%Two random 4 x 4 matrices:
A = rand(4,4);
B = rand(4,4);

%An elementary subtraction of rows (by left-multiplying. Of columns if you right-)
E = [1 -3  0  0;
     0  1  0  0;
     0  0  1  0;
     0  0  0  1];

%An "odd" permutation:
P_odd = [1  0  0  0;
         0  1  0  0;
         0  0  0  1;
         0  0  1  0];

%An "even" permutation:
P_even = [0  1  0  0;
          1  0  0  0;
          0  0  0  1;
          0  0  1  0];

%Another 4 x 4...almost, the 1st row is missing:
C = rand(3,4);

%Two random row vectors
a1 = rand(1,4);
a2 = rand (1,4);

%Two matrices having the a_1, a_2 as 1st rows
D1 = [a1;
      C ]

D2 = [a2;
      C ]
%Matrix with sum as 1st row
D = [a1 + a2;
        C  ]
```

Now, using these matrices do the following tests. We've slipped in a couple of *false* ones - to make it more exciting (take a guess before you hit < `enter` >).

(a) `det(D1) + det(D2) = det(D)`

(b) `det(A) + det(B) = det(A + B)`

(c) `det(10 * A) = 10 * det(A)`

(d) `det(E * A) = det(A) = det(A * E)`

(e) `det(P_odd * A) = -det(A)`
`det(P_even * A) = det(A)`

Which ones in (a)-(e) are correct, and which are false?

Solution. (a) True

(b) False

(c) False

(d) True

(e) True

□

18.06 Wisdom. Enjoyed your spring break? True!