

18.06 Spring 2012 – Problem Set 5

This problem set is due Thursday, March 22nd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 11 & 16 from Section 4.2.

Solution.

Problem 11

(a)

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Hence

$$p = A\hat{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

(b)

$$AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Hence

$$p = A\hat{x} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 16

Linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ which is closest to b is the projection of b onto the plane spanned by $(1, 2, -1)$ and $(1, 0, 1)$.

projection of b onto the plane

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $(2, 1, 1)$ itself is on the plane and $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. \square

2. Do Problem 27 from Section 4.2.

Solution. The vector Ax is in the nullspace of A^T . Ax is always in the column space of A . To be in both of those perpendicular spaces, Ax must be zero. \square

3. Do Problem 30 from Section 4.2.

Solution. (a) Column space of A is spanned by one vector $a = [3, 4]^T$. The projection matrix

$$P_c = \frac{aa^T}{a^T a} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}.$$

(b) The row space of A is spanned by one vector $b = [1, 2, 2]^T$. The projection matrix

$$P_r = \frac{bb^T}{b^T b} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

$B = P_c A P_r = A$. Since P_c projects each columns of A to themselves, $P_c A = A$. Similarly, $P_r A^T = A^T$ and $A = (P_r A^T)^T = A P_r^T = A P_r$. Hence $P_c A P_r = A$.

\square

4. Do Problem 31 from Section 4.2.

Solution. p is the projection of b onto the subspace spanned by a_1, \dots, a_n if and only if $b - p$ is perpendicular to the vectors a_1, \dots, a_n . \square

5. Do Problems 1 & 3 from Section 4.3.

Solution. **Problem 1**

We have

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \hat{x} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

by substituting this matrix to the normal equation $A^T A \hat{x} = A^T v$. Hence

$$\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, A \hat{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}.$$

The minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = 44$.

Problem 3 Since $eA = 0$, e is perpendicular to both columns of A . The shortest distance $\|e\|$ from b to the column space of A is \sqrt{E} for the E in Problem 1. Hence it is $\sqrt{44}$.

□

6. Do Problem 26 from Section 4.3.

Solution. We would like to minimize the error in the equation of the form $Ax = b$ given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}.$$

Although this equation is not solvable, we solve a new equation of the form $A\hat{x} = p$, where p is the projection of b to $C(A)$. Recall that $p = A(A^T A)^{-1} A^T b = (1/2, 1/2, 7/2, 7/2)$. Solving $A\hat{x} = p$ then yields $\hat{x} = (2, -3/2, -3/2)$ [NOTE: These solutions earlier said $\hat{x} = (2, -1, -3/2)$, in error].

Then our plane of best fit is given by the equation $2 - \frac{3}{2}x - \frac{3}{2}y = 0$, and at the centre of the square (i.e. $x = y = 0$), the plane passes through $(0, 0, 2)$. 2 is the average of 0, 1, 3, 4.

□

7. Do Problems 4 & 14 from Section 4.4.

Solution.

Problem 4

(a) Take $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I$.

(b) $(0, 0)$ and $(1, 0)$ are orthogonal since their dot product is zero. But they are not independent.

(c) One possible set of orthonormal basis including q_1 is

$$\begin{aligned} q_1 &= (1, 1, 1)/\sqrt{3} \\ q_2 &= (1, -1, 0)/\sqrt{2} \\ q_3 &= (1, 1, -2)/\sqrt{6}. \end{aligned}$$

Problem 14

$B = b - xa$ should be orthogonal to a , with $x = \frac{a^T b}{a^T a} = 2$, so $B = (2, -2)$. Then $q_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $q_2 = \frac{1}{2\sqrt{2}}(2, -2) = (1/\sqrt{2}, -1/\sqrt{2})$. Then

$$QR = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}.$$

□

8. Do Problem 7 from Section 4.4.

Solution. We normally solve $A^T A \hat{x} = A^T b$ to get the least squares solution \hat{x} . Now, if Q is an orthonormal matrix, we have $Q^T Q = I$, so that we just get $\hat{x} = Q^T b$. □

9. Do Problems 18 & 24 from Section 4.4.

Solution. Problem 18

Note that we are only looking for orthogonal vectors, not orthonormal ones. So we have

$$A = a = (1, -1, 0, 0),$$

$$B = b - \text{projection of } b \text{ onto } A = (1/2, 1/2, -1, 0),$$

$$C = c - \text{projection of } c \text{ onto } A - \text{projection of } c \text{ onto } B = (1/3, 1/3, 1/3, -1).$$

Problem 24

- (a) A possible basis is given by $(1, -1, 0, 0)$, $(1, 0, -1, 0)$, $(1, 0, 0, 1)$.
- (b) Since S contains solutions to $(1, 1, 1, -1)^T x = 0$, a basis of S is given by $(1, 1, 1, -1)$.
- (c) Split $(1, 1, 1, 1) = b_1 + b_2$ by projection onto S^\perp and S : $b_2 = 1/2(1, 1, 1, -1)$ and $b_1 = 1/2(1, 1, 1, 3)$.

□

10. In this exercise, we use MATLAB for learning about Gram-Schmidt.

Suppose the 4×3 matrix A has 1's when $i = j$ and -1 's when $i = j + 1$ and otherwise $A_{ij} = 0$. From the MATLAB command

$$[Q, R] = \text{qr}(A),$$

find the Gram-Schmidt orthonormal basis for the column space $C(A)$. Renormalize those basis vectors to contain nice fractions by dividing by the diagonal entries of R . Now the vectors are orthogonal, not orthonormal.

Guess what the pattern you see here is, and use this to find a 4th vector in \mathbb{R}^4 orthogonal to these three columns of Q (and of A).

Solution. >>A = [1 0 0; -1 1 0; 0 -1 1; 0 0 1]

A =

```
1 0 0
-1 1 0
0 -1 1
0 0 -1
```

>>[Q,R] = qr(A)

Q =

```
-0.7071 -0.4082 -0.2887 0.5000
 0.7071 -0.4082 -0.2887 0.5000
        0  0.8165 -0.2887 0.5000
        0         0  0.8660 0.5000
```

R =

```
-1.4142  0.7071      0
         0 -1.2247  0.8165
         0         0 -1.1547
         0         0      0
```

So an orthonormal basis of $C(A)$ is given by $(-0.7071, 0.7071, 0, 0)$, $(-0.4082, -0.4082, 0.8165, 0)$, and $(-0.2887, -0.2887, -0.2887, 0.8660)$.

From $R = \begin{pmatrix} -1.4142 & 0 & 0 & 0 \\ 0 & -1.2247 & 0 & 0 \\ 0 & -1.1547 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the orthogonal ma-

trix is given by $QD = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/2 \\ -1 & 1/2 & 1/3 & 1/2 \\ 0 & -1 & 1/3 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$. where D is the 4×4 diagonal matrix

with the diagonal entries being $(-1.4142, -1.2247, -1.1547, 1)$.

If we were to find the fourth vector orthogonal to all the columns of Q , we would guess $(1/4, 1/4, 1/4, 1/4)$. One can check that this is indeed perpendicular to all the columns.

Remark 0.1. When we use the `qr` command, MATLAB always returns a square matrix Q with orthonormal columns (in our case, we only needed three orthonormal vectors in \mathbb{R}^4 , but MATLAB finds a fourth vector that is orthonormal to the first three, and returns it as the last column of Q). In order to avoid this issue, you could have used the command `[Q,R] = qr(A,0)`. Then it would return:

Q =

-0.7071	-0.4082	-0.2887
0.7071	-0.4082	-0.2887
0	0.8165	-0.2887
0	0	-0.8660

R =

-1.4142	0.7071	0
0	-1.2247	0.8165
0	0	-1.1547

Sorry about the confusion!

□

18.06 Wisdom. On problem sets and exams, always try to think of ways to check your results. It both assists your own learning, when you reflect on and relate to what you've worked through, plus you avoid losing points!