18.06 Spring 2012 – Problem Set 4

This problem set is due Thursday, March 8th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 5 from Section 3.5.

   Solution. (a) The vectors are independent. One way to see this is to arrange the vectors as columns of a matrix, do elimination, and observe that there are 3 pivots.

   (b) The vectors are dependent. One way to see this is to observe that they sum to zero.

   □

2. Do Problem 7 from Section 3.5.

   Solution. To show that the differences are dependent, observe that $\mathbf{v_1} + \mathbf{v_3} - \mathbf{v_2} = \mathbf{0}$. In the equation

$$\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{bmatrix} = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

   the matrix $\begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix}$ is invertible and the other two matrices are singular. □

3. Do Problems 15 & 18 from Section 3.5.

   Solution. Solution to 3.5.15 If $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are linearly independent, the space they span has dimension n. These vectors are a basis for that space. If the vectors are columns of an m by n matrix, then m is greater than or equal to n. If $m = n$, that matrix is invertible *Solution to 3.5.18*

   (a) might not

   (b) are not

   (c) might be

   □

4. Do Problem 28 from Section 3.5.

Solution. The following is a basis for the space of 2 by 3 matrices where the entries of each column add to zero:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The following is a basis for the subspace of the above where the entries of each row also add to zero:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

□

5. Do Problem 45 from Section 3.5.

Solution. Take a basis $\mathbf{v_1}, \ldots, \mathbf{v_k}$ for $\mathbf{V}$ and a basis $\mathbf{w_1}, \ldots, \mathbf{w_l}$ for $\mathbf{W}$. We know that

$$k + l = dim(\mathbf{V}) + dim(\mathbf{W}) > n$$

Therefore, there exists a nontrivial linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{w_1}, \ldots, \mathbf{w_l}$ that gives zero; let this combination be

$$\sum a_i \mathbf{v_i} + \sum b_j \mathbf{w_j}$$

Then the vector

$$\sum a_i \mathbf{v_i} = -\sum b_j \mathbf{w_j}$$

is a non-zero vector that is in both $\mathbf{V}$ and $\mathbf{W}$. □

6. Do Problem 4 from Section 3.6.

Solution.

(a) We can just take the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Its column space clearly contains $(1, 1, 0)$ and $(0, 0, 1)$ and its row space contains the standard basis vectors $(1, 0)$ and $(0, 1)$ for $\mathbb{R}^2$, so the row space contains *all* vectors in $\mathbb{R}^2$.

(b) For the column space and the null space both to be in $\mathbb{R}^3$ we need a 3×3 matrix. But then the dimensions of the column space and the null space must add to 3, so they can't both be 1-dimensional.

(c) Take the matrix

$$\begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Its null space is all of $\mathbb{R}^2$ and its left null space is all of $\mathbb{R}$.

(d) Consider the matrix

$$\begin{pmatrix} 9 & 3 \\ -3 & -1 \end{pmatrix}.$$

Its row space clearly contains $(3, 1)$, and

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 9 & 3 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

(e) This is impossible — we know that the left null space is the space of vectors orthogonal to the column space and the null space is the space of vectors orthogonal to the row space. Therefore if the row space and the column space are the same, the null space and the left null space must also be the same. □

7. Do Problem 9 from Section 3.6.

Solution. (a) The row spaces of $[A]$ and $\left[\begin{smallmatrix} A \\ A \end{smallmatrix}\right]$ are clearly the same, since the rows of the second matrix are the same as those of the first, just repeated. Since the null space consists of the vectors orthogonal to the row space, they also have the same null space. Since the rank equals the dimension of the row space, they have the same rank.

(b) The column spaces of $\left[\begin{smallmatrix} A \\ A \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right]$ are clearly the same, since the columns of the second matrix are the same as those of the first, just repeated. Since the left null space consists of the vectors orthogonal to the column space, they also have the same left null space. Since the rank equals the dimension of the column space, they have the same rank, which by (a) is also the rank of $[A]$. □

8. Do Problem 16 from Section 3.6.

Solution. If $\mathbf{v}$ is a row of A then it is orthogonal to every vector in the null space. But the only vector that is orthogonal to itself is the zero vector; since $\mathbf{v} \neq \mathbf{0}$ it therefore can't be in the null space if it is a row. □

9. Do Problem 18 from Section 3.6.

Solution. The combination (row 3) - 2(row 2) + (row 1) is zero. Therefore $(1, -2, 1)$ is in the left null space. Since we see from the echelon form that A has rank 2 (there are two pivot columns), the null space has dimension 1 and so consists of vectors of the form $c \cdots (1, -2, 1)$ for all scalars c. From the echelon form we can also see that (column 3) - 2(column 2) + (column 1) is zero, so $(1, -2, 1)$ is also in the null space. Since the rank of A is 2 its null space has dimension 1 too, so for this matrix the null space and the left null space happen to be the same. □

10. Do Problem 32 from Section 3.6.

Solution. Since A and B have the same column space, the two identity matrices must have the same size, say $k \times k$. Therefore the matrices F and G have the same size too. Because A and B have the same row spaces, the ith row of A must be a linear combination of the rows of B. We can clearly ignore the rows that are all zeros, so we have a linear combination a_1(row 1) + a_2(row 2) $+ \cdots +$ a_k(row k). But the first k components of this row vector are $(a_1, a_2, \ldots, a_k)$, so to get the ith row of A we must have $a_i = 1$ and $a_j = 0$ for $j \neq i$. In other words, the ith row of B equals the ith row of A. □

18.06 Wisdom. The first exam is coming up (next Friday, March 9th, Walker Memorial, 11am-12 noon). It will include concepts - not only numbers. Your most valuable study resources will be the book, the problems you've solved so far and the old exams (with solutions!) on the course web: `http://web.mit.edu/18.06/www/old.shtml`. If we were you, we would spend this weekend brushing up on concepts and trying to solve as many old exams as possible.